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D-Brane Anomaly Inflow Revisited

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Abstract

Axial and gravitational anomaly of field theories, when embedded in string theory, must be accompanied by canceling inflow. We give a self-contained overview for various world-volume theories, and clarify the role of smeared magnetic sources in I-brane/D-brane cases. The proper anomaly descent of the source, as demanded by regularity of RR field strengths H's, turns out to be an essential ingredient. We show how this allows correct inflow to be generated for all such theories, including self-dual cases, and also that the mechanism is now insensitive to the choice between the two related but inequivalent forms of D-brane Chern-Simons couplings. In particular, $SO(6)_R$ axial anomaly of $d = 4$ maximal SYM is canceled by the inflow onto D3-branes via the standard minimal coupling to C_4. We also propose how, for the anomaly cancelation, the four types of Orientifold planes should be coupled to the spacetime curvatures, of which conflicting claims existed previously.

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1   Introduction

Field theories with extended supersymmetries are equipped with R-symmetry and sometimes other accidental global symmetries, which become typically anomalous at one-loop. In a slight abuse of nomenclature, we will call them collectively the axial anomaly. While the axial anomaly is not a consistency issue at the level of field theories, it becomes one when one realizes such a theory as a part of string theory or M-theory. In the latter context, the R-symmetry and the global symmetries are realized as a part of ten- or eleven-dimensional diffeomorphism invariance, whose anomaly will render the system gravitationally inconsistent, so there has to be a canceling contribution from the underlying string theory or M-theory. Clearly the gravitational anomaly belongs to the same class, so must be considered simultaneously with the axial anomaly. For many field theories arising as world-volume dynamics of M-branes and D-branes, the anomaly inflow has been cataloged and shown to cancel the one-loop anomaly of the field theory.

The anomaly inflow originates from the topological couplings and resulting modification of the Bianchi identities of anti-symmetric tensor fields. A prime example of this is the Green-Schwarz mechanism for type I and Heterotic string theory. Other than this, there are two principal systems where the anomaly inflow is important. One class is the M5-brane, and the others are the D-branes and I-branes (meaning intersection of D-branes). For the M5-branes, inflow occurs from a spacetime topological coupling to a lower dimensional world-volume, and can be understood more easily as a direct result of the modified Bianchi identity [1][2][3]. As far as anomaly inflow goes, M5-brane represents the best understood example, although its world-volume field theory, namely $(2,0)$ theory, and in particular how the one-loop anomaly arises remain largely mysterious.

On the other hand, the easiest examples of world-volume field theories with one-loop axial/gravitational anomaly are maximally supersymmetric Yang-Mills theories in $d = 4, 6, 8, 10$ dimensions, some of which can be realized on coincident D-branes. As we review in section 2, anomalies in d dimensions are dictated by topological $(d+2)$-form polynomials, which, for the maximally supersymmetric Yang-Mills theories, are[#1]

$$(-1)^{d/2}\pi \cdot ([ch_{\mathrm{adj}}(\mathcal{F}) + l] \wedge \mathcal{A}(R) \wedge [ch_{S^+}(F_R) - ch_{S^-}(F_R)])\Big|_{(d+2)-form}, \qquad (1.1)$$

with the Chern class ch, the A-roof genus $\mathcal{A}$, gauge field strength $\mathcal{F}$, spacetime curvature tensor R, and the number of $U(1)$ factors l in the gauge group [6]. F_R is

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[#1] For $d = 4k$, one actually has $-2ch_{S_-}(F_R)$ in place of $[ch_{S_+}(F_R) - ch_{S_-}(F_R)]$, but this substitution does not affect the relevant $(4k+2)$-form part.

the field strength of the external R-gauge field and $S_\pm$ are the chiral and anti-chiral spinor representations of the R-symmetry $SO(10-d)_R$. We are using a Lorentzian signature $(-++\cdots+)$.[#2]

With $G = U(n)$, $SO(n)$, $Sp(n)$, this is a world-volume theory of Dp-branes with or without various Orientifold planes. As such, the gravitational curvature R and the R-symmetry curvature F_R are, respectively, associated with the tangent bundle $\mathcal{T}$ and the normal bundle $\mathcal{N}$ of the world-volume and we will henceforth rewrite the anomaly polynomial as

$$I_{d=p+1}^{1-loop} = (-1)^{(p+1)/2}\pi \cdot ([ch_{\mathrm{adj}}(\mathcal{F}) + l] \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})]) \Big|_{(d+2)-form},$$
(1.2)

where $l = 1$ for $U(n)$ and 0 otherwise. As such, the anomaly must be canceled by other stringy contributions, since the diffeomorphism invariance must be preserved. This anomaly is null for $d = 2$, so we expect a canceling inflow for $4 \le d \le 10$.

For cancelation of such anomaly, one must understand inflow onto D-branes, which is a little more involved than the M5 case. One reason is that the relevant topological coupling (with $2\pi\sqrt{\alpha'} = 1$), say, for each stacks of coincident Dp-branes with charge μ_p,

$$S_{CS} = \frac{\mu_p}{2} \int_{Dp} \sum_{r \le p} s^*(C_{r+1}) \wedge ch(\mathcal{F}) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}},$$
(1.3)

lives in the same world-volume as the relevant one-loop anomaly. Throughout this paper, we denote the pull-back of spacetime forms to the relevant world-volume by s^*. The overall factor $1/2$ [4] may be a little puzzling, but is a consequence of the so-called duality symmetric formulation of C's. The latter is necessary because electromagnetic dual pairs of RR fields always act together in I-brane/D-brane inflow mechanism. Most importantly, this factor $1/2$ disappears in the field equation derived from the duality symmetric formulation, which resolves all the potential conflicts and, in particular, helps one to recover the Dirac quantization conditions with the usual, properly quantized charges μ_p.[#3]

A somewhat unexpected result in literatures, though, is that, under the standard procedure, one finds the right canceling inflow only if the following alternative and inequivalent form of these couplings is used [4][5],

$$\frac{\mu_p}{2} \int_{Dp} \left(N_p\, s^*(C_{p+1}) \pm \sum_{r<p} s^*(H_{r+2}) \wedge (\cdots) \right),$$
(1.4)

[#2]Here we took the computations of Ref. [6] in the Euclidean signature $(+++\cdots+)$, and weak-rotated it to $(-++\cdots+)$. This produces the overall sign in (1.1).

[#3]See Appendices A and B.

where H is the gauge-invariant field strengths of C and N_p is the number of coincident Dp-branes. The ellipsis represents the odd-form Chern-Simons densities from $ch(\mathcal{F})\mathcal{A}(\mathcal{T})^{1/2}\mathcal{A}(\mathcal{N})^{-1/2}$. See section 2.3 and equation (2.15) for complete details. The two would be equivalent if $H = dC$, but this is not the case because $dH \neq 0$ in general. This failure of the Bianchi identity is the main mechanism that underlies the inflow, and one traditionally finds two different inflow from these two sets of couplings. Furthermore, the case of D3-branes proved to be fairly subtle among these examples. The one-loop anomaly polynomial (1.1) reduces for $d = 4$ to

$$\dim G \times \frac{1}{24\pi^2} \operatorname{tr}_{S^+} F_R^3 \, , \qquad (1.5)$$

and is purely axial with $SO(6)_R$ R-symmetry. $\dim G$ is the dimension of the gauge group G, and other terms cancel out thanks to the reality of the adjoint representation. Yet, the conventional procedure involving the above topological couplings on D-branes fails to generate any inflow at all for D3-branes which, in view of how various D-branes are connected to each other by T-dualities, sounds quite odd.

In this note, we wish to revisit these anomaly inflow and clarify some of these finer points. We will emphasize on how we must regularize the Bianchi identity and magnetic sources. The regularization is not necessary for the simplest type of inflow, such as gravitational anomaly of M5-brane theory. For others, regularization is essential. For the axial anomaly of M5-branes, this has been exploited carefully by Freed, Harvey, Minasian, and Moore (FHMM) [3], while it also played some role in the Cheung-Yin's (CY) [4] elaboration of I-brane inflow arguments by Green, Harvey, and Moore (GHM) [5]. At the end of day, however, several unsatisfactory aspects remain, one of which is the apparent absence of D3 anomaly inflow we already mentioned. In this note, we combine the ideas of FHMM and of GHM/CY to address the D-brane and I-brane anomaly inflow again and resolve such outstanding issues.

In section 2 and 3, we review various anomaly inflow mechanisms in string theory and in M-theory. In section 2, after a brief review of the consistent anomaly and the simplest inflow mechanism (gravitational anomaly on a M5-brane), we retrace the steps taken by CY for D-brane and I-brane inflow. In particular, we note that, to obtain the desired inflow, they had to use the modified Chern-Simons couplings (1.4) rather than the more natural looking one (1.3) [4]. We will delineate how the usual procedure also fails to produce necessary D3-brane anomaly inflow. In section 3, we turn to the anomaly inflow onto M5-branes associated with the $SO(5)_R$ symmetry by FHMM. Although the mechanism of inflow here is qualitatively different from other examples, we will learn an important lesson that should be applied to the D-brane and I-brane story.

In section 4, we reconsider the D-brane and I-brane inflow by requiring both source terms in the Bianchi identity and the RR field strengths to be regular, which

are of course interconnected to each other. This requirement modifies the solution to the Bianchi identity, in a manner that fundamentally changes gauge transformation properties of the RR gauge fields. With this revised transformation rule, we re-derive the anomaly inflow for D-branes and I-branes, and find that the standard Chern-Simons coupling of type (1.3) generates all the necessary anomaly inflow, as well as (1.4). In particular, this includes "self-dual" cases like the D3-branes. Despite the naive difficulties with these "self-dual" cases, the correct inflow arises without a special treatment.

In section 5, we extend all these discussion to systems involving Orientifold planes. In literature, there appears to be partially conflicting claims regarding what should be the right (gravitational) Chern-Simons couplings on the four types of Orientifold planes [11][12][13][14][15]. Here we settle this by requiring cancelation of the axial and gravitational anomaly of orthogonal and symplectic gauge theories and also demanding that the inflow to be canceled by closed string one-loop contribution is independent of the Orientifold type.

2   Anomaly Inflows

2.1   Consistent Anomaly

Recall [16] that the so-called consistent anomaly on d dimensions is represented by a characteristic polynomial of rank $d + 2$, say $X(F, R, \cdots)$, of curvature tensors, via a descent relation,

$$X_{d+2} = dX_{d+1}^{(0)} \ , \qquad \delta X_{d+1}^{(0)} = dX_d^{(1)} \ , \tag{2.1}$$

such that the anomaly associated with X_{d+2} is actually an integral of $X_d^{(1)}$. Note that this procedure is ambiguous since $X_{d+1}^{(0)} \rightarrow X_{d+1}^{(0)} + dZ_d$ with d-form Z_d. However, this is not an issue because the additional anomaly due to this shift is δZ_d and thus cancelable by a local counter-term $-Z_d$. This simple observation gives us a useful generality about anomaly: when $X = Y \wedge \tilde{Y}$, the anomaly due to X can be expressed as

$$\beta Y \wedge \tilde{Y}^{(1)} + (1 - \beta) Y^{(1)} \wedge \tilde{Y} \ , \tag{2.2}$$

since a counter term of type $Y^{(0)} \wedge \tilde{Y}^{(0)}$, provided that both the factors exist, can always shift the parameter β. In this note, we will mostly use the symmetric version $\beta = 1/2$.

It is important to note that when one of the two factors, say $\tilde{Y}$, is 0-form and thus constant, we must use $Y^{(1)}\tilde{Y}$ and vice versa. More generally, when Y includes a

0-form constant, say $Y = Y_0 + dY^{(0)}$ etc, we have instead

$$(Y \wedge \tilde{Y})^{(1)} = Y_0 \tilde{Y}^{(1)} + Y^{(1)} \tilde{Y}_0 + (dY^{(0)} \wedge \tilde{Y}^{(1)} + Y^{(1)} \wedge d\tilde{Y}^{(0)})/2 , \qquad (2.3)$$

which is different from $(Y \wedge \tilde{Y}^{(1)} + Y^{(1)} \wedge \tilde{Y})/2$.

In usual field theories, anomaly arises from one-loop. In Fujikawa's path integral formulation, this can be understood as a failure of the path integral measure to respect symmetry of the Lagrangian. There are also situations where such an anomaly is present at tree level. The Wess-Zumino-Witten term of chiral perturbation theory [17], which captures one-loop flavor anomaly of QCD, is probably one of the oldest such example. This type of tree-level anomaly in low energy effective action is there because the 't Hooft anomaly matching condition must be respected. Another important class of tree-level anomaly is called the anomaly inflow, in string theory setting, which arises to cancel would-be harmless anomaly associated with global symmetries of a field theory because the global symmetries are typically gauged once embedded in string theory. Regardless of precise mechanism of how it is generated, however, anomaly can be cast into the above form as a descent from a characteristic class of rank $d + 2$.

2.2 M5-Brane Gravitational Anomaly Inflow

Perhaps the simplest example of such an inflow can be found in the context of a single M5-brane, whose world-volume theory is a tensor multiplet theory in six dimensions. To set a consistent convention, let us write the 11-dimensional supergravity action as

$$S_{11} = \frac{1}{2\kappa_{11}^2} \left[\int \sqrt{-g} \left(R - \frac{1}{2} |G_4|^2 \right) - \frac{1}{6} \int C_3 \wedge G_4 \wedge G_4 \right] + \mu_{M2} \int C_3 \wedge I_8 , \quad (2.4)$$

with the three-form gauge field of M-theory C_3, its field strength G_4, M2-brane tension μ_{M2}, and the 8-form polynomial I_8 of the spacetime curvature two-form

$$I_8 = -\frac{1}{48} \left(p_2(R) - \frac{1}{4} p_1(R)^2 \right) , \qquad (2.5)$$

where p_n's are the Pontryagin classes. See Appendix C for definition of characteristic classes we will encounter in this note. In terms of the 11-dimensional Planck length l_p, recall that

$$\frac{1}{2\kappa_{11}^2} = \frac{2\pi}{(2\pi l_p)^9} , \quad \mu_{M2} = \frac{2\pi}{(2\pi l_p)^3} , \quad \mu_{M5} = \frac{2\pi}{(2\pi l_p)^6} . \qquad (2.6)$$

For the M5-brane anomaly inflow discussion, we are using the unit $2\pi l_p = 1$, whereby $1/2\kappa_{11}^2 = \mu_{M2} = \mu_{M5} = 2\pi$.

In the presence of a M5, which couples to C_3 magnetically,

$$dG_4 = 2\kappa_{11}^2 \mu_{M5} \delta_{M5} = \delta_{M5} \ , \tag{2.7}$$

this coupling induces a tree-level anomaly on the M5 world-volume. The argument starts with the alternate form of the topological coupling

$$\mu_{M2} \int C_3 \wedge I_8 \quad \rightarrow \quad \mu_{M2} \int G_4 \wedge I_7^{(0)} \ , \qquad dI_7^{(0)} = I_8 \ , \tag{2.8}$$

which varies under the eleven dimensional diffeomorphism [1] as

$$\mu_{M2} \int G_4 \wedge \delta I_7^{(0)} = -\mu_{M2} \int dG_4 \wedge I_6^{(1)} = -2\pi \int_{M5} I_6^{(1)} \ , \tag{2.9}$$

as $2\kappa_{11}^2 \mu_{M2} \mu_{M5} = 2\pi$. This inflow is capable of canceling world-volume anomaly of the form,

$$2\pi I_8(\mathcal{T} \oplus \mathcal{N}) = -2\pi \times \frac{1}{48}\left(p_2(\mathcal{T}) + p_2(\mathcal{N}) - \frac{(p_1(\mathcal{T}) - p_1(\mathcal{N}))^2}{4} \right) \ , \tag{2.10}$$

where $\mathcal{T}$ and $\mathcal{N}$ denote tangent and normal bundles of the M5-brane.

On the other hand, the one-loop anomaly polynomial of a single tensor multiplet is

$$2\pi \mathcal{J}_8 = -2\pi \times \frac{1}{48}\left(p_2(\mathcal{T}) - p_2(\mathcal{N}) - \frac{(p_1(\mathcal{T}) - p_1(\mathcal{N}))^2}{4} \right) \ . \tag{2.11}$$

If we concentrate on the gravitational anomaly, encoded in $\mathcal{T}$, the inflow above completely cancels the one-loop contribution. When we consider n M5-branes, this inflow grows linearly with n, and so is capable of canceling the gravitational anomaly from n tensor multiplets, also.

However, as is clear from the above, the cancelation is not actually complete when we consider the axial anomaly as well. Inflow $-I_8$ plus the one-loop anomaly $\mathcal{J}_8$ leave

$$2\pi(\mathcal{J}_8 - I_8(\mathcal{T} \oplus \mathcal{N})) = 2\pi \times \frac{1}{24} p_2(\mathcal{N}) \tag{2.12}$$

uncanceled [2]. We will come back to how this remaining axial anomaly is canceled shortly, as this mechanism is more subtle and its variant will be needed to clarify the D-brane anomaly inflow in the next section. For now, let us first consider a slightly different anomaly inflow to D-branes, where the bilinear and quadratic inflows can be generated.

2.3 I-Brane/D-Brane Inflow

A more involved example of the anomaly inflow arises in the D-brane context. The axial and gravitational anomaly are quite prevalent and in fact most supersymmetric Yang-Mills theories with $d \geq 4$ have such anomalies. Many of these theories are realizable as world-volume theories from D-branes and Orientifold planes, whereby one must ask what are the analog of the above anomaly inflow mechanism for D-branes. On Dp-branes, there are well-known topological couplings between Ramond-Ramond tensor fields and the spacetime curvature. In fact, these coupling (modulo the normal bundle part) was conjectured initially [5] by asking that the anomaly associated with bi-fundamental hypermultiplets along the intersections of two types of D-branes, with world-volumes M_1 and M_2 respectively. Each set of D-branes carry the above couplings, which can induce inflow onto the intersection $N = M_1 \cap M_2$ and cancel anomaly due to the bi-fundamental fields there. This is what is known as the I-brane anomaly inflow, with "I" signifying the intersection of D-branes. We will shortly review how this works in a more general setting, including the case of $N = M_1 = M_2$ [4], for which case we refer to the D-brane inflow.

To produce the right anomaly inflow, one usually starts with the world-volume topological coupling [4],

$$\sum_A (S'_{CS})^A \,, \tag{2.13}$$

where we summed over stacks of coincident D-branes, labeled by A, with the revised topological coupling alluded to in $(1.4)^{\#4}$

$$S'_{CS} = \frac{\mu_p}{2} \int_{Dp} \left(s^*(C_{p+1}) \wedge Y_0 + (-1)^\epsilon \sum_{r<p} s^*(H_{r+2}) \wedge Y^{(0)}_{p-r-1} \right) \,, \tag{2.15}$$

with $\epsilon = 0, 1$ for type IIA/IIB branes and $H = dC + \cdots$. We will keep track of the different stacks, by labeling various world-volume objects, such as s^* or Y's by the label A. So, for example the characteristic classes are defined as

$$Y_n^A \equiv \left[ch(\mathcal{F}_A) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T}_A)}{\mathcal{A}(\mathcal{N}_A)}} \right]\Bigg|_n = \delta_n^0 \cdot Y_0^A + (1 - \delta_n^0) \cdot d(Y^A)^{(0)}_{n-1} \,, \tag{2.16}$$

while the corresponding Chern-Simons densities $(Y^A)^{(0)}$ are defined by $d((Y^A)^{(0)}) = Y^A$. The world-volume gauge field strength $\mathcal{F}_A$ is in the fundamental representation

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#4With our choice of unit $2\pi\sqrt{\alpha'} = 1$, for D-brane discussions, $\mu_p = 2\pi/(4\pi^2\alpha')^{(p+1)/2} = 2\pi$ and $2\kappa_{10}^2 = (4\pi^2\alpha')^4/2\pi = 1/2\pi$. The curvature tensors in the topological couplings then have the standard normalization,

$$ch(\mathcal{F}) = \operatorname{tr} e^{\mathcal{F}/2\pi} \tag{2.14}$$

and so on.

of $U(Y_0^A = N_p^A)$. The orientation of the world-volume will be declared later when we discuss the equation of motion for C's.

Note that this form of the Chern-Simons couplings differ from the natural world-volume topological couplings (1.3), and, as we will see shortly, the two generate two different inflow even though the shifted Bianchi identities are the same: Seemingly, only S'_{CS} generates the right inflow to cancel the world-volume one-loop anomaly, which we will later attributes to mishandling of the Bianchi identity.

Note that the unfamiliar but crucial factor $1/2$ in front of the coupling. This reflects the subtlety [19][4] that we must include C_s and their magnetic dual C_{8-s} on equal footing. In order for this to make sense, the accompanying kinetic action for the RR fields must be written in a way that does not distinguish electric and magnetic fields, which effectively absorbs half of the usual minimal couplings. An important consistency check is that this factor $1/2$ does not appear in the field equations and Bianchi identities. See Appendix A for a toy example that illustrates how this is achieved, and Appendix B for precise form of the RR field kinetic terms.

The equation of motion that follows from this coupling is

$$d\left(*(H_{r+2})\right) = -(-1)^r \sum_B 2\kappa_{10}^2 \mu_q \, Y_{q-r}^B \wedge \Delta_{9-q}^B \, , \tag{2.17}$$

with some "delta function" $(9-q)$-form, Δ_{9-q}^B, representing the D-brane position. Because this is not a scalar object, however, the expression becomes ill-defined unless we carefully regularize and covariantize it. This smearing of the magnetic source is a recurring and necessary step when we discuss the anomaly inflow, especially when the anomaly associated with normal bundle needs to be discussed. Thus, we write instead,

$$d\left(*(H_{r+2})\right) = -(-1)^r \sum_B 2\kappa_{10}^2 \mu_q \, Y_{q-r}^B \wedge \tau_{9-q}^B \, , \tag{2.18}$$

where we smeared the sources due to the Dq-branes by introducing a "delta-function" $(9-q)$-form τ_{9-q}^B, well-identified in the mathematical literatures as the Thom class of the normal bundle $\mathcal{N}$ [18]. See Appendices A and B for detailed derivations.

We will study it in more detail later, but it suffices to note here the general form,

$$\tau_{9-q} = d(\rho \, \hat{e}_{8-q}) \, . \tag{2.19}$$

The "radial" function ρ, whose support determines the smearing of the source, interpolates between -1 on the brane and 0 at infinity. The global angular form $\hat{e}_{8-q}$ is essentially a covariantized volume-form, normalized to unit volume, of a $(8-q)$-sphere surrounding the Dp-brane. In particular $\delta\hat{e}_{8-q} = 0$, and $d\hat{e}_{8-q} = 0$ for even q and $d\hat{e}_{8-q} = -\chi(\mathcal{N})_{9-q}$ with the Euler class χ for odd q. By choosing ρ to have increasingly small support near the origin, we can localize the source with arbitrary

precision, and with diffeomorphism invariance preserved. In addition we will also choose $\rho'(0) = 0$. With arbitrary small support of ρ, we can take Y's to be uniform along the normal direction, which allows (2.18) to make sense.

Since this equation of motion exists for all C_{q+1}'s, it also implies, with $*H_n = (-1)^{(n-2+\epsilon)/2}H_{10-n}$, the modified Bianchi identities

$$dH_{8-r} = -\sum_B 2\kappa_{10}^2 \mu_q (-1)^{(-q+\epsilon)/2} \wedge \bar{Y}_{q-r}^B \wedge \tau_{9-q}^B , \qquad (2.20)$$

with $\bar{Y}$'s being the complex conjugated Y's,

$$\bar{Y}_n^A = \left[ch(-\mathcal{F}_A) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T}_A)}{\mathcal{A}(\mathcal{N}_A)}} \right]\Bigg|_n . \qquad (2.21)$$

We note here again that this shifted Bianchi implies that S_{CS} of (1.3) and S'_{CS} of (2.15) are not equivalent. From this, CY noted the following solution to the Bianchi

$$H_{8-r} = d(C_{7-r}) - \sum_B 2\kappa_{10}^2 \mu_q (-1)^{(-q+\epsilon)/2}(\bar{Y}^B)_{q-r-1}^{(0)} \wedge \tau_{9-q}^B , \qquad (2.22)$$

and that gauge-invariance of the field strength is ensured if they allow C's to be gauge-variant as

$$\tilde{\delta}C_{7-r} = \sum_B 2\kappa_{10}^2 \mu_q (-1)^{(-q+\epsilon)/2}(\bar{Y}^B)_{q-r-2}^{(1)} \wedge \tau_{9-q}^B , \qquad (2.23)$$

where $\tilde{\delta}$ denotes the gauge transformation here, to distinguish it against the revised one in section 4. Thus S'_{CS} is gauge-variant and generates tree-level anomaly,

$$\tilde{\delta}S'_{CS} = \frac{1}{2} \sum_A \mu_p \int_A \left(s_A^*(\tilde{\delta}C_{p+1})Y_0^A + (-1)^\epsilon \sum_{r<p} s_A^*(H_{r+2}) \wedge d(Y^A)_{p-r-2}^{(1)} \right) . \qquad (2.24)$$

This is CY's master formula to the I-brane inflow, which has been used to cancel many of known one-loop anomalies for field theories on the intersecting brane. When we consider a pair of intersecting D-brane stacks, this can cancel the anomaly from the bi-fundamental fermions propagating along the intersection, in particular, which is the origin of the name I-brane inflow.

A special case of this discussion occurs for a single stack of Dp-branes with $p = 5, 7$.[5] The world-volume theories would be the maximally supersymmetric Yang-Mills theories in $d = 6, 8$, whose one-loop anomaly polynomial is given in (1.2). The

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[5] $p = 4, 6, 8$ are also acceptable, except that the relevant field theories are of odd dimensions and neither one-loop anomaly nor anomaly inflow is generated. $p = 3$ appears difficult since the product of two τ's will give 12-forms and thus vanishes against the spacetime integration. However, as we will see later, this comes from mishandling of the Thom class in this context.

gauge variations itself involves a factor of τ_{9-p}, which must be pulled-back to the world-volume defined by a limit of the same τ_{9-p}. While naively this looks like an ill-defined procedure, this is not so because all the troublesome pieces in τ_{9-p} actually vanishes upon the pull-back, s^*, and the only surviving piece is

$$s^*(\tau_{9-p}) = s^*(d\rho \wedge \hat{e}_{8-p} - \rho \cdot \chi(\mathcal{N})_{9-p}) = \chi(\mathcal{N})_{9-p} , \qquad (2.25)$$

where we used $\rho'(0) = 0$ and $\rho(0) = -1$. That is, the pull-back of the Thom class to the zero section equals the Euler class [18].

With $1/2 \times 2\kappa_{10}^2 \mu_p^2 = \pi$, the anomaly inflow $\tilde{\delta}S'_{CS}$ from the self-intersection of these Dp-branes is then

$$(-1)^{(-p+1)/2}\pi \qquad (2.26)$$
$$\times \int_{\mathrm{D}p} \left((\bar{Y})^{(1)}_{2p-8} Y_0 + \sum_{6-p<r<p} \bar{Y}_{p+r-6} \wedge (Y^{(1)})_{p-r-2} + \bar{Y}_0 Y^{(1)}_{2p-8} \right) ,$$

which equals, up to local counter terms,

$$(-1)^{(-p+1)/2}\pi \int_{\mathrm{D}p} \left(\sum_{r\geq 0} Y_r \wedge \sum_{s\geq 0} \bar{Y}_s \right)^{(1)}_{2p-8} \wedge \chi(\mathcal{N})_{9-p} . \qquad (2.27)$$

Using the definition of Y and $\bar{Y}$'s and also $ch^{SU(n)}_{\mathrm{adj}}(\mathcal{F}) + 1 = ch(\mathcal{F})ch(-\mathcal{F})$ for $U(n)$ gauge group, we find

$$\tilde{\delta}S'_{CS} = -(-1)^{(p+1)/2}\pi \int_{\mathrm{D}p} \left([ch^{SU(n)}_{\mathrm{adj}}(\mathcal{F}) + 1] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \right)^{(1)} \wedge \chi(\mathcal{N}) . \quad (2.28)$$

When $p \geq 5$, this is equivalent to, again up to local counter-terms,[6]

$$\tilde{\delta}S'_{CS} = -(-1)^{(p+1)/2}\pi \int_{\mathrm{D}p} \left([ch^{SU(n)}_{\mathrm{adj}}(\mathcal{F}) + 1] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N}) \right)^{(1)} , \quad (2.29)$$

and equals, upon the identity $\chi(\mathcal{N})\mathcal{A}(\mathcal{N})^{-1} = ch_+(\mathcal{N}) - ch_-(\mathcal{N})$,

$$= -(-1)^{(p+1)/2}\pi \int_{\mathrm{D}p} \left([ch^{SU(n)}_{\mathrm{adj}}(\mathcal{F}) + 1] \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_+(\mathcal{N}) - ch_-(\mathcal{N})] \right)^{(1)} . \quad (2.30)$$

which has precisely the right form to cancel the one-loop anomaly (1.2) of the maximally supersymmetric $U(n)$ Yang-Mills theory in the respective dimensions.

---

[6] This last step works because for $p \geq 5$ the 0-form part of the characteristic classes, ch and $\mathcal{A}$, are irrelevant upon integration; χ is a $(9-p)$-form and the integration is over $(p+1) > (9-p)$ dimensions.

By the way, the overall sign is not related to whether we are considering Dp's or anti-Dp's. For a single stack, an extra overall sign in the coupling of D-branes to C's, cancels out when we put back $\tilde{\delta}C$ to compute variation of S'_{CS}. From one-loop perspective, this happens because, as we flip the chirality of world-volume fermions, their representations under $SO(9-p)$ R-symmetry also flip as the fermions have a definite ten-dimensional chirality; The sign flip from the chirality flip is canceled by the exchange of $S_+(\mathcal{N})$ and $S_-(\mathcal{N})$ representations. As was mentioned in footnote #2, this overall sign appears to be associated with the canonical choice of the chirality operator and the accompanying signature $(-++\cdots+)$, relative to those chosen in Ref. [6].

There are a couple of unsatisfactory issues that remain here. One problem, as mentioned several times already, concerns the case of $p = 3$, which apparently produces no inflow. This is due to $S^*(\tau_6) = \chi_6$ in the inflow formula, since a 6-form integrates to zero against the four world-volume dimensions. The one-loop axial anomaly is nontrivial, and something else must compensate for the anomaly, yet it is difficult to imagine D3-branes, despite their self-dual nature, can be that different.

The second issue, which is a little more of technical nature, is that, to produce the correct inflow for generic cases, one must use S'_{CS} instead of S_{CS}. Although the resulting Bianchi identity is the same, the action themselves are not equivalent, and one finds

$$\tilde{\delta}S'_{CS} \neq \tilde{\delta}S_{CS} \ , \tag{2.31}$$

even up to local counter terms. If we started with S_{CS} and followed the same procedure as above, we would have arrived at an analog of (2.26) effectively without the last term in the parenthesis there. S_{CS} in (1.3) looks far more natural, but does not yield the right canceling inflow even for $p = 5, 7$. Both of these curiosities were noted by Cheung and Yin [4].

As we will see in section 4, these two problems have a common origin and is solved by more careful treatments of the regularized source τ's.

3   M5-Brane Axial Anomaly Inflow

A slightly simpler version of this last issue has been discussed in the context of M5-brane normal bundle anomaly, so we will review this first. Recall that, after the anomaly inflow from $G_4 \wedge I_7^{(0)}$ term onto a M5-brane, we have a leftover

$$2\pi \times \frac{1}{24}\, p_2(\mathcal{N}) \ , \tag{3.1}$$

as shown in (2.12). Further cancelation of this is more subtle and known to originate from a revised version of the spacetime Chern-Simons coupling

$$- \frac{2\pi}{6} \int C_3 \wedge G_4 \wedge G_4 \ , \tag{3.2}$$

upon careful regularization of the C_3 [3]. Here we follow FHMM [3] almost verbatim.[#7]

To obtain the right inflow, let us recall the Bianchi identity (2.7) (with $2\pi l_p = 1$)

$$dG_4 = \delta(y^1) \cdots \delta(y^5) dy^1 \cdots dy^5 \ , \tag{3.3}$$

with normal bundle coordinate y^i's. As in the I-brane/D-brane discussion, this expression needs modification if we wish to be careful about the normal bundle part. We should substitute the right hand side with a covariant and smeared version of the delta function, namely the Thom class

$$dG_4 = \tau_{M5}(\mathcal{N}) \ , \tag{3.4}$$

which can be written as before

$$\tau_{M5} = d[\rho(r) \wedge \hat{e}_4] = d\rho \wedge \hat{e}_4 \ . \tag{3.5}$$

$\rho(r)$ is a smooth function of radial direction, with $d\rho$ serving as a smoothed radial delta-function satisfying $\rho(r) = -1$ on the M5-brane and $\rho(r) = 0$ far from the branes. $\hat{e}_4$ is a global angular form, which is closed as the normal bundle is of odd dimension.[#8]

More explicitly, we have

$$
\begin{aligned}
\hat{e}_4(\Theta) &= \frac{1}{64\pi^2} \epsilon_{a_1 \cdots a_5} [(D\hat{y})^{a_1} (D\hat{y})^{a_2} (D\hat{y})^{a_3} (D\hat{y})^{a_4} \hat{y}^{a_5} \\
&\quad -2 F^{a_1 a_2} (D\hat{y})^{a_3} (D\hat{y})^{a_4} y^{a_5} + F^{a_1 a_2} F^{a_3 a_4} \hat{y}^{a_5}] \ ,
\end{aligned} \tag{3.6}
$$

with

$$(Dy)^a = d\hat{y}^a - \Theta^{ab} \hat{y}^b \ , \quad F^{ab} = d\Theta^{ab} - \Theta^{ac} \wedge \Theta^{cb} \ , \tag{3.7}$$

[#7]Except for renaming the global angular form as

$$\hat{e}_4 = (e_4/2)_{FHMM}$$

and clarification of a related normalization issue.

[#8] An interesting attempt to assign a microscopical origin of such a smearing, albeit in a toy model, can be found in Ref. [21].

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in terms of $SO(5)$ connection $\Theta^{ab} = -\Theta^{ba}$ and the normalized Cartesian coordinates $\hat{y}^a = y^a/r$ along the fibre. Using $\delta\Theta^{a_1 a_2} = (D\Lambda)^{a_1 a_2}$ and $\delta\hat{y}^a = \Lambda^{a_1 a_2}\hat{y}^{a_2}$, the descents of e_4,

$$\hat{e}_4 = d\hat{e}_3^{(0)}, \quad \delta\hat{e}_3^{(0)} = d\hat{e}_2^{(1)}, \tag{3.8}$$

are

$$
\begin{aligned}
\hat{e}_3^{(0)}(\Theta) &= \frac{1}{32\pi^2}\epsilon_{a_1\cdots a_5}[\Theta^{a_1 a_2}d\Theta^{a_3 a_4}\hat{y}^{a_5} \\
&\quad -\frac{1}{2}\Theta^{a_1 a_2}\Theta^{a_3 a_4}d\hat{y}^{a_5} - 2\Theta^{a_1 a_2}d\hat{y}^{a_3}d\hat{y}^{a_4}\hat{y}^{a_5}] ,
\end{aligned}
\tag{3.9}
$$

and

$$\hat{e}_2^{(1)}(\Lambda,\Theta) = \frac{1}{16\pi^2}\epsilon_{a_1\cdots a_5}[\Lambda^{a_1 a_2}d\hat{y}^{a_3}d\hat{y}^{a_4}\hat{y}^{a_5} - \Lambda^{a_1 a_2}\Theta^{a_3 a_4}d\hat{y}^{a_5}] . \tag{3.10}$$

Now we can solve the Bianchi identity (3.5),

$$G_4 = dC_3 + [\beta\rho\hat{e}_4 - (1-\beta)d\rho \wedge \hat{e}_3^{(0)}] , \tag{3.11}$$

with arbitrary real number β. Note that ρe_4 diverges at the origin, since integral over any arbitrary small four-sphere around the origin gives a finite value. On the other hand, $d\rho \wedge \hat{e}_3^{(0)}$ can be managed to be finite near the origin, by requiring $d\rho \to 0$ as $r \to 0$. Hence we should choose $\beta = 0$, to ensure the regularity of C_3 and G_4 near the M5-branes, so

$$G_4 = dC_3 - d\rho \wedge \hat{e}_3^{(0)} . \tag{3.12}$$

The solution implies that C_3 transforms nontrivially under the $SO(5)_R$ gauge transformation in order that G_4 is invariant:

$$\delta C_3 = -d\rho \wedge \hat{e}_2^{(1)} . \tag{3.13}$$

In the presence of such M5 sources, the Chern-Simons term

$$-\frac{2\pi}{6}\int_{M_{11}} C_3 \wedge G_4 \wedge G_4 , \tag{3.14}$$

becomes ambiguous. FHMM suggested that the right modification is to replace C_3 by $C_3 - \sigma_3 \equiv C_3 - \rho\hat{e}_3^{(0)}$ and G_4 by $G_4 - \rho\hat{e}_4$ with the properties,

$$
\begin{aligned}
G_4 - \rho\hat{e}_4 &= d(C_3 - \sigma_3) , \\
\delta(C_3 - \sigma_3) &= d(-\rho \cdot \hat{e}_2^{(1)}) .
\end{aligned}
\tag{3.15}
$$

The modified Chern-Simons term

$$S'_{CS} = -\frac{2\pi}{6}\lim_{\epsilon\to 0}\int_{M_{11}-D_\epsilon(M5)} (C_3 - \sigma_3) \wedge d(C_3 - \sigma_3) \wedge d(C_3 - \sigma_3) , \tag{3.16}$$

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where we subtract the infinitesimal tubular neighborhood, $D_\epsilon(M5)$, of the world-volume $M5$ with arbitrary small radius ϵ. Its gauge-variation is

$$\delta S'_{CS} = \frac{2\pi}{6} \lim_{\epsilon \to 0} \int_{M_{11} - D_\epsilon(M5)} d(\rho \cdot \hat{e}_2^{(1)}) \wedge d(C_3 - \sigma_3) \wedge d(C_3 - \sigma_3) \; . \qquad (3.17)$$

of which C_3 parts vanish with $\epsilon \to 0$. The integrand is a well-defined total derivative, so we are left with an integral over the sphere bundle $S_\epsilon(M5)$ of vanishing radius

$$\delta S'_{CS} = -\frac{2\pi}{6} \int_{S_\epsilon(M5)} \hat{e}_4 \wedge \hat{e}_4 \wedge \hat{e}_2^{(1)} = -2\pi \int_{M5} \frac{p_2(\mathcal{N})^{(1)}}{24} \; , \qquad (3.18)$$

which neatly cancels the normal bundle anomaly that was left over in section 2.1.

The two anomaly inflows to the M5-brane can each be generalized easily to the case of N coincident branes. One inflow is linear in C while the other is cubic in C, so the total inflow has to be

$$-2\pi \left(N \times I_8(\mathcal{T} \oplus \mathcal{N}) + N^3 \times \frac{1}{24} p_2(\mathcal{N}) \right) \; , \qquad (3.19)$$

from which we infer the world-volume one-loop anomaly of A_{n-1} $(2,0)$ theory plus a free tensor multiplet theory as

$$2\pi \left((N-1) \times \mathcal{J}_8(\mathcal{T}, \mathcal{N}) + (N^3 - N) \times \frac{1}{24} p_2(\mathcal{N}) \right) + 2\pi \mathcal{J}_8(\mathcal{T}, \mathcal{N}) \; . \qquad (3.20)$$

Of course this shows the famous N^3 scaling of $(2,0)$ theories [22][23][24][25].

4   D-Brane Anomaly Inflow Revisited

The salient point we wish to learn from the M5-brane axial anomaly inflow is how, in (3.12), FHMM solved the Bianchi identity in the presence of a smeared delta function source in the form of the Thom class. Even though the Thom class on the M5-brane was introduced as $d(\rho \cdot \hat{e}_4)$, the descent formula $\tau_{M5} = d\tau_{M5}^{(0)}$ was written as

$$\tau_{M5}^{(0)} = -d\rho \wedge \hat{e}_3^{(0)} \; , \qquad (4.1)$$

$\tau_{M5}^{(1)}$ of which at the end generated the necessary inflow for the M5-brane axial anomaly. The argument in favor of $-d\rho \wedge \hat{e}_3^{(0)}$ as the unique choice (instead of $\rho \cdot \hat{e}_4$) is that, since one resolved the magnetic source into a smooth configuration, the field strength should remain smooth everywhere and in particular on the M5-brane. With $\rho(0) = -1$, $\rho \hat{e}_4$ is singular and ill-defined at the origin, whereas $-d\rho \wedge \hat{e}_3^{(0)}$ is regular at the origin as long as we choose $\rho'(0) = 0$.

Note that, in the discussion of I-brane/D-brane inflow, the shifted Bianchi identity (2.18) also involved Thom classes $\tau_{9-q} = d(\rho \cdot \hat{e}_{8-q})$ for the Dq-branes, but was solved as

$$H_{s+2} = \cdots - \sum_B 2\kappa_{10}^2 \mu_q \, (-1)^{(-q+\epsilon)/2} (\bar{Y}^B)_{q+s-7}^{(0)} \wedge \tau_{9-q}^B \; . \tag{4.2}$$

Note that the descent of τ_{9-q} is apparently not invoked. Actually, for terms with $q + s = 6$, for which $\bar{Y}$ factor is a number, $\tau_{8-q}^{(0)}$ must appear on the right hand side, since otherwise the Bianchi identity is not obeyed. Presumably this term is suppressed because the obvious choice $\tau^{(0)} = \rho \cdot \hat{e}$ is gauge-invariant and seemingly irrelevant for the inflow. However, $\rho \cdot \hat{e}$ is neither a unique choice for descent nor physically sensible. With $\rho = -1$, $H_{s+2} \sim \cdots + \bar{Y}_0 \cdot \rho \cdot \hat{e}_{s+2} + \cdots$ would be singular and ill-defined at the origin. In this section, we will address this problem and study the ramifications.

We wish to emphasize here that we will be using, instead of S_{CS}' of (2.15), the natural Chern-Simons coupling (1.3) which is

$$S_{CS} = \frac{\mu_p}{2} \int_{Dp} \sum_{r \leq p} s^*(C_{r+1}) \wedge Y_{p-r} \; , \tag{4.3}$$

with $Y = ch(\mathcal{F})\mathcal{A}(\mathcal{T})^{1/2}\mathcal{A}(\mathcal{N})^{-1/2}$. As usual, s^* is the pull-back to the world-volume (i.e., to the zero section of the normal bundle). With the revised inflow mechanism below, S_{CS} and S_{CS}' will be shown to generate the equivalent anomaly inflow in the end.

4.1   Revised Inflow from Regularity of H

First, we need to clarify an important difference between the Thom classes of even and odd dimensional bundles. For odd fibre dimensions (applicable to even q and thus type IIA branes), $\tau_{9-q} = d(\rho \cdot \hat{e}_{8-q})$ behaves in much the same way as τ_{M5} of the previous section. For even fibre dimensions (applicable to odd q and thus type IIB branes), the global angular form decomposes into two pieces [18][20]

$$\hat{e}_{8-q} = v_{8-q} + \Omega_{8-q}(\mathcal{N}) \; , \tag{4.4}$$

where the first term involves at least one normal vector field $\hat{y}$ and can be written locally as

$$v_{8-q} = d\psi_{7-q} \; , \tag{4.5}$$

while the last term is nothing but the Chern-Simons term of the Euler class with a sign flip, i.e.,

$$d\Omega_{8-q}(\mathcal{N}) = -\chi(\mathcal{N})_{9-q} \; . \tag{4.6}$$

Clearly, this behavior of the Thom class is responsible, with $\rho(0) = -1$, for the identity $s^*(\tau) = \chi$. Finally the gauge-invariance of $\hat{e}$ implies that

$$\delta\psi_{7-q} = -\Omega^{(1)}_{7-q} = \chi(\mathcal{N})^{(1)}_{7-q} \ . \tag{4.7}$$

Ω exists for even-dimensional normal bundles, and so this is relevant for all type IIB branes.

Note that v_{8-q} (and its descent ψ_{7-q}) is singular at the origin, being a normalized volume form of S^{8-q}. In contrast, $\Omega(\mathcal{N})_{8-q}$ is composed only of the gauge fields of the normal bundle and is well-defined and smooth everywhere. For regular solutions of H, we must then choose the following descent for τ,

$$\tau^{(0)}_{8-q} = -d\rho \wedge \psi_{7-q} + \rho \cdot \Omega_{8-q} \ , \tag{4.8}$$

which results in

$$\tau^{(1)}_{7-q} = -\rho \cdot \chi(\mathcal{N})^{(1)}_{7-q} \ . \tag{4.9}$$

Note that both expressions are regular at the origin, with $\rho'(0) = 0$. This gives

$$H_{s+2} = d(C_{s+1}) - \sum_B 2\kappa^2_{10}\mu_q \, (-1)^{(-q+\epsilon)/2} \left(\bar{Y}^B \wedge \tau^B\right)^{(0)}_{s+2} \ , \tag{4.10}$$

where, for type IIB theory,

$$\left(\bar{Y}^B \wedge \tau^B\right)^{(0)}_{s+2} = \beta(\bar{Y}^B)^{(0)}_{q+s-7} \wedge \tau^B_{9-q} + (1-\beta)(\bar{Y}^B)_{q+s-6} \wedge \left(-d\rho \wedge \psi_{7-q} + \rho \cdot \Omega_{8-q}\right)^B \ . \tag{4.11}$$

Although β is an arbitrary real number in general, we must take $\beta = 0$ when $\bar{Y}$ on the left hand side is a 0-form (here, $q + s = 6$). Its gauge variation gives

$$\left(\bar{Y}^B \wedge \tau^B\right)^{(1)}_{s+1} = \beta(\bar{Y}^B)^{(1)}_{q+s-8} \wedge \tau^B_{9-q} + (1-\beta)\bar{Y}^B_{q+s-6} \wedge \left(-\rho \cdot \chi^{(1)}_{7-q}\right)^B \ . \tag{4.12}$$

With this understood, the gauge transformation of C is,

$$\delta C_{s+1} = \sum_B 2\kappa^2_{10}\mu_q \, (-1)^{(-q+\epsilon)/2} \left(\bar{Y}^B \wedge \tau^B\right)^{(1)}_{s+1} \ . \tag{4.13}$$

Note the difference from (2.23). The difference is essential for terms with $\bar{Y}_{0=q+s-6}$ factor and its consequence in δS_{CS} below cannot be removed by local counter-terms.

Let us concentrate on the case of a single stack of type IIB Dp-branes. The gauge variation of S_{CS} (4.3) is

$$\delta S_{CS} = (-1)^{(-p+1)/2}\pi \int_{Dp} \sum_r s^* \left(\left(\bar{Y}_{p+r-6} \wedge \tau_{9-p}\right)^{(1)}\right) \wedge Y_{p-r} \ . \tag{4.14}$$

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Just as $s^*(\tau) = \chi$, it is easy to show that

$$s^*(\tau^{(1)}) = s^*(-\rho\chi^{(1)}) = \chi^{(1)} \ , \tag{4.15}$$

and that

$$\delta S_{CS} = (-1)^{(-p+1)/2}\pi \int_{Dp} \sum_r \left(\bar{Y}_{p+r-6} \wedge \chi_{9-p}(\mathcal{N})\right)^{(1)} \wedge Y_{p-r} \ , \tag{4.16}$$

which equals

$$-(-1)^{(p+1)/2}\pi \int_{\mathrm{D}p} \left(ch(-\mathcal{F}) \wedge \sqrt{\overline{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}}} \wedge \chi(\mathcal{N})\right)^{(1)} \wedge \left(ch(\mathcal{F}) \wedge \sqrt{\overline{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}}}\right) \ . \tag{4.17}$$

With $p < 9$, $\chi(\mathcal{N})_{9-p}$ is never 0-form, allowing us to rewrite this as, up to local counter terms,[#9]

$$\begin{aligned} \delta S_{CS} &= -(-1)^{(p+1)/2}\pi \int_{\mathrm{D}p} \left(ch(\mathcal{F}) \wedge ch(-\mathcal{F}) \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N})\right)^{(1)} \tag{4.18} \\ &= -(-1)^{(p+1)/2}\pi \int_{\mathrm{D}p} \left([ch_{\mathrm{adj}}^{SU(n)}(\mathcal{F}) + 1] \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_+(\mathcal{N}) - ch_-(\mathcal{N})]\right)^{(1)} \ . \end{aligned}$$

Of these, for $p = 1$, the expression is null and no inflow is generated. For others, $p = 3, 5, 7$, this is precisely the right inflow to cancel one-loop anomaly (1.2) for $d = 4, 6, 8$. ($p = 9$ requires a special treatment as it always involves an Orientifold plane. See section 5.)

We have re-analyzed the Bianchi identities of RR field strengths by requiring the regularity of physical variables. This is not by a choice but required, since the D-brane inflow analysis must have the magnetic sources regulated anyway. Singular field strengths in the absence of singular source do not make any sense. Although the final answer looks superficially the same as before, it differs in two important aspects and addresses the concerns raised at the end of section 2.2. First, the revised inflow is now applicable for all D-branes and I-branes. In particular, it produces right answers for a single stack of Dp-branes including $p = 3$ case while the old procedure produced a null result for $p = 3$ and produced right answers for $p \geq 5$ only. Second, with the revised gauge transformation rule, $\delta \neq \tilde{\delta}$, the natural Chern-Simons couplings S_{CS} (1.3) generates the correct anomaly inflow. Furthermore, it is not difficult to see that, *up to local counter terms on world-volume,*

$$\delta S_{CS} = \delta S'_{CS} \ , \tag{4.19}$$

so the unreasonable sensitivity to the precise form of Chern-Simons couplings, as in (2.31), is no longer there.

[#9]$p = 9$ requires a separate discussion since this case involves Orientifold planes. See next section.

4.2 Axial Anomaly Inflow onto D3-Branes

In particular, this resolves a long-standing issue regarding D3-brane anomaly inflow. Our analysis shows that the correct inflow arises also for D3-branes; one-loop anomaly of the maximal $U(N_3)$ super Yang-Mills theory is completely canceled by the anomaly inflow onto N_3 coincident D3-branes. Previous analysis produced a null inflow for this case, seemingly requiring another inflow mechanism [4]. The crucial difference between the old and the revised inflow is whether one has a 6-form $s^*(\tau_6) = \chi_6$ as a blind overall factor (which kills off all terms) or one also has an exceptional term with 4-form $s^*(\tau_4^{(1)}) = \chi_4^{(1)}$ instead. Here we wish to retrace the case of D3-branes, with more care given to details of the Thom class, for a pedagogical reason.

Upon close inspection of the inflow, one can see easily that, for Dp-branes, only those RR gauge fields from C_{p+1} down to its dual C_{7-p} contribute to the inflow. For an N_3 coincident D3, C_4 is self-dual, and the only relevant term for D3-brane inflow is the minimal coupling[#10]

$$S_{CS}^{D3} = \frac{\mu_3 N_3}{2} \int_{D3} s^*(C_4) \ , \qquad (4.20)$$

with the constant 0-form $Y_0 = N_3 = \bar{Y}_0$. This is also related to the fact that $s^*(\tau^{(1)}) = \chi^{(1)}$ is already a 4-form, saturating all the world-volume dimensions. From this, combining with the self-duality constraint on C_4, we have the Bianchi identity of H_5

$$dH_5 = 2\kappa_{10}^2 \mu_3 N_3 \tau_6(D3) \ , \qquad (4.21)$$

again with the regularized and covariantized $\tau_6(D3)$.

Recall that this Thom class is defined by

$$\tau_6(D3) = d(\rho \cdot \hat{e}_5) \ , \qquad (4.22)$$

with the global angular five-form $\hat{e}_5$ of unit volume. More explicitly,

$$\begin{aligned}
\hat{e}_5 \ = \ & -\frac{1}{15}\epsilon_{a_1\cdots a_6} D\hat{y}^{a_1} D\hat{y}^{a_2} D\hat{y}^{a_3} D\hat{y}^{a_4} D\hat{y}^{a_5} \hat{y}^{a_6} \\
& -\frac{1}{6}\epsilon_{a_1\cdots a_6} F_R^{a_1 a_2} D\hat{y}^{a_3} D\hat{y}^{a_4} D\hat{y}^{a_5} \hat{y}^{a_6} - \frac{1}{8}\epsilon_{a_1\cdots a_6} F_R^{a_1 a_2} F_R^{a_3 a_4} D\hat{y}^{a_5} \hat{y}^{a_6} \ , \ (4.23)
\end{aligned}$$

which can be decomposed as

$$\hat{e}_5 = d\psi_4 + \Omega_5 \ , \qquad (4.24)$$

[#10] Since D3 is a self-dual object, one might wonder whether this electric-type minimal coupling suffices. See Appendix B for why this is so when one uses the duality symmetric formulation for RR-gauge field kinetic term. The latter in particular imposes the self-duality condition for C_4 as an equation of motion, with or without the minimal coupling to C_4.

with

$$d\psi_4 = -\frac{1}{120\pi^3}\,\epsilon_{a_1\cdots a_6}d\hat{y}^{a_1}d\hat{y}^{a_2}\cdots d\hat{y}^{a_5}\hat{y}^{a_6} + \cdots \,, \qquad (4.25)$$

and

$$\Omega_5 = \frac{1}{384\pi^3}\,\epsilon_{a_1\cdots a_6}\left[F_R^{a_1 a_2}F_R^{a_3 a_4}A_R^{a_5 a_6} + \cdots\right]\,, \qquad d\Omega_5 = -\chi_6(F_R)\,. \qquad (4.26)$$

Of course the six-form χ_6 and the five-form Ω_5 vanish identically when evaluated on the four dimensional world-volume of D3, but what matters at the end is the appearance of the 4-form $\chi_4^{(1)}$ from the variation of ψ_4. In what follows, we obtain the same final answer if we remove χ_6 and Ω_5 from all the formulae but remember that $\delta\psi_4$ is trivially closed on the D3 world-volume.

As before, from the regularity requirement of H_5 and C_4, we must choose among many naive choices of $[\tau_6(D3)]^{(0)}$,

$$H_5 = dC_4 + 2\kappa_{10}^2\mu_3 N_3\,(\tau(D3))_5^{(0)} = dC_4 + 2\kappa_{10}^2\mu_3 N_3\left[\rho \wedge \hat{e}_5 - d(\rho \wedge \psi_4)\right]\,. \qquad (4.27)$$

On the other hand, since

$$\delta\hat{e}_5 = 0\,, \qquad \delta\psi_4 = \chi_4^{(1)}\,, \qquad (4.28)$$

the gauge invariance of H_5 yields

$$s^*(\delta C_4) = -2\kappa_{10}^2\mu_3 N_3 \times s^*\left(\tau_6(D3)^{(1)}\right) = -2\kappa_{10}^2\mu_3 N_3 \times \chi_4^{(1)}\,. \qquad (4.29)$$

If we substitute this to δS_{CS}^{D3}, we finally have

$$\delta S_{CS}^{D3} = -\kappa_{10}^2\mu_3^2 N_3^2 \int_{D3}\chi_4^{(1)} = N_3^2 \times \left(-\pi\int_{D3}\chi_4^{(1)}\right)\,, \qquad (4.30)$$

with $\kappa_{10}^2\mu_3^2 = [(2\pi)^7(\alpha')^4/2] \times [1/(2\pi)^3(\alpha')^2]^2 = \pi$. This cancels exactly the one-loop anomaly on the D3-branes.

As we saw in the introduction, the $SO(6)_R$ axial anomaly polynomial at one-loop of the $U(N_3)$ theory is

$$\begin{aligned} I_6 &= \frac{N_3^2}{24\pi^2}\,\mathrm{tr}_{S^+}F_R^3 = N_3^2 \cdot 2\pi \cdot ch_{S^+}(F_R)\Big|_{6-form} \\ &= N_3^2 \cdot \pi \cdot [ch_{S^+}(F_R) - ch_{S^-}(F_R)]\Big|_{6-form}\,, \qquad (4.31) \end{aligned}$$

where F_R is the curvature tensor of an external $SO(6)_R$ in the Weyl representation. The bracket in the last line equals the Euler class divided by the A-roof genus, and the Euler class is already 6-form, so the one-loop anomaly polynomial equals

$$I_6 = N_3^2 \times \pi\chi(F_R)\,, \qquad (4.32)$$

which is precisely canceled by the inflow (4.30).

The case of D3 is special in that the minimal coupling to C_4 alone generates the anomaly inflow and there is no need to invoke lower-rank RR gauge fields. This happens due to the self-dual nature of D3. A toy model of such self-dual objects, namely dyonic string in six dimensions, was studied previously in Refs. [26][27][28]. Our inflow argument is related most directly to that of Ref. [27]. There is also some relation to Ref. [26] in that $v_5 = d\psi_4$ is the generalization of the Wess-Zumino-Witten term of the latter, but the inflow here is a direct consequence of the standard topological coupling, rather than with additional modifications. In particular, the smearing function ρ plays a crucial role here.

5 Chern-Simons Couplings on Orientifold Planes

Extending all of these to the presence of Orientifold planes should be straightforward. The main extra ingredient is how the various Orientifold planes couple to the space-time curvature. For Op^- plane, the relevant Chern-Simons coupling is known to be,

$$S_{Op^-} = \frac{1}{2} \times \left(-2^{p-4}\frac{\mu_p}{2} \int_{Op^-} \sum_r s^*(C_{r+1}) \wedge \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \right) \, , \qquad (5.1)$$

where $\mathcal{L}$ is the Hirzebruch class [7][8][9][10]. There are various studies in the past that worked out analog of this for other three classes of Orientifold planes, but the answers seem to disagree partially with one another [11][12][13][14][15].

In this section, we will show that the one-loop anomaly from the gauge sector cancels away by the anomaly inflow, if we assume the most obvious choices of the Orientifold Chern-Simons couplings, which in addition to the above O$^-$,

$$S_{\widetilde{Op^-}} = \frac{1}{2} \times \left(-\frac{\mu_p}{2} \int_{\widetilde{Op^-}} \sum_r s^*(C_{r+1}) \wedge \left[\sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} - 2^{p-4}\sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \right] \right) \, , \qquad (5.2)$$

reflecting the usual statement that this case has a single, unpaired D-brane stuck at the Orientifold plane. For Op^+,

$$S_{Op^+} = \frac{1}{2} \times \left(2^{p-4}\frac{\mu_p}{2} \int_{Op^+} \sum_r s^*(C_{r+1}) \wedge \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \right) \, , \qquad (5.3)$$

and the same expression for $S_{\widetilde{Op^+}}$. This last one associated with symplectic type orbifolding agrees with Refs. [11][12].

As before, the overall factor $1/2$ exists only when we write the kinetic terms of RR tensors in the duality symmetric form, and does not enter the equation of motion. The other $1/2$ factor accompanying μ_p is due to the Orientifolding projection.

5.1   Dp-Op Inflow

We work in the covering space of the Orientifold and take care to divide by two at the end of everything. For example, equation of motion and the Bianchi identity are unaffected by this, but the action written in the covering space must be either divided by two (e.g., world-volume part) or restricted to the half space (e.g., spacetime part). Similarly, the D-brane Chern-Simons couplings are

$$S_{Dp} = \frac{1}{2} \times \left(\frac{1}{2} \sum_A \mu_p \int_A \sum_{r \leq p} s^*(C_{r+1}) \wedge ch_{2k}(\mathcal{F}_A) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T}_A)}{\mathcal{A}(\mathcal{N}_A)}} \right) . \tag{5.4}$$

Note that here we assumed these $2k$ Dp branes are on the top of the Op^- plane, so they share the Thom class τ, the tangent bundle $\mathcal{T}$, and the normal bundle $\mathcal{N}$.

Note that, upon the Orientifold projection, some of the RR tensor fields are absent. With Op planes, $C_{p-1\pm 4n}$ maps to its negative and thus are projected out, while $C_{p+1\pm 4n}$ remains intact. This can potentially modify inflow argument. However, we do not really lose any term since $ch_{2k}(\mathcal{F})$ is a sum of $4n$-forms for $SO(2k)$ and $Sp(k)$ gauge groups, and since the Euler character χ_{9-p} is a $(9-p)$-form monomial. An exception to this is $p = 9$, for which one of the relevant RR gauge field, $C_{10=9+1}$, does not exist, and $Y_{10}^{(1)}$ type of inflow cannot be generated. This is precisely what leads to the tadpole condition $2k = 32$ for type I string theory. See section (5.4) for a separate review of D9-O9.

With this, we may proceed as before except that $Y = ch_{2k}(\mathcal{F})\mathcal{A}(\mathcal{T})^{1/2}\mathcal{A}(\mathcal{N})^{-1/2}$ is shifted by -2^{p-4} times

$$Z \equiv \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} , \tag{5.5}$$

and the Bianchi identity reads

$$d(H_{s+2}) = -\sum_B 2\kappa_{10}^2 \mu_q (-1)^{(-q+\epsilon)/2} (\bar{Y}_{q+s-6}^B - 2^{p-4} \bar{Z}_{q-r}^B) \wedge \tau_{9-p}^B , \tag{5.6}$$

from which we repeat the procedure of sec.4.1 and arrive at the world-volume expressions,

$$\delta(S_{Op^-} + S_{Dp}) = -(-1)^{(p+1)/2} \cdot \frac{\pi}{2} \int \left(\bar{Y} \wedge Y \wedge \chi(\mathcal{N}) \right)^{(1)}$$

$$+(-1)^{(p+1)/2} \cdot \frac{\pi}{2} \cdot 2^{p-4} \int \left((\bar{Y} \wedge Z + \bar{Z} \wedge Y) \wedge \chi(\mathcal{N}) \right)^{(1)}$$

$$-(-1)^{(p+1)/2} \cdot \frac{\pi}{2} \cdot 2^{2(p-4)} \int \left(\bar{Z} \wedge Z \wedge \chi(\mathcal{N}) \right)^{(1)}$$

$$\equiv (-1)^{(p+1)/2} \int \left(\Delta_{BB} + \Delta_{BO^-+O^-B} + \Delta_{O^-O^-} \right) , \qquad (5.7)$$

where in the last line we classified the contribution to brane-brane(BB), brane-plane(BO), and plane-plane(OO) type.

Again we denote by ch_ρ the trace over ρ representation of $SO(2k)$. In particular, $ch_{2k} = ch_{\overline{2k}}$ and $ch_{2k\otimes\overline{2k}} = ch_{2k\otimes 2k} = [ch_{2k}]^2$, thanks to the reality of the vector representation of SO groups. Then, we find contributions with gauge group factors

$$\Delta_{BB} = -\frac{\pi}{2} \left(ch_{2k\otimes\overline{2k}}(\mathcal{F}) \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} , \qquad (5.8)$$

and[#11]

$$\begin{aligned}
\Delta_{BO^-+O^-B} &= \frac{\pi}{2} \cdot 2^{p-4} \left([ch_{2k}(\mathcal{F}) + ch_{\overline{2k}}(\mathcal{F})] \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} \wedge \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} \\
&= \frac{\pi}{2} \cdot 2^{p-4} \left([ch_{2k}(\mathcal{F}) + ch_{\overline{2k}}(\mathcal{F})] \wedge \frac{\mathcal{A}(\mathcal{T}/2)}{\mathcal{A}(\mathcal{N}/2)} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} \\
&= \frac{\pi}{2} \left(ch_{2k}(2\mathcal{F}) \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} ,
\end{aligned}$$

which combine to

$$\begin{aligned}
&(-1)^{(p+1)/2} \left(\Delta_{BB} + \Delta_{BO^-+O^-B} \right) \\
&= -(-1)^{(p+1)/2} \left(\frac{\pi}{2} [ch_{2k\otimes 2k}(\mathcal{F}) - ch_{2k}(2\mathcal{F})] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} . \qquad (5.9)
\end{aligned}$$

Purely Orientifold contribution is

$$\begin{aligned}
(-1)^{(p+1)/2} \Delta_{O^-O^-} &= -(-1)^{(p+1)/2} \frac{\pi}{2} \cdot 2^{2(p-4)} \left(\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} \\
&= -(-1)^{(p+1)/2} \frac{\pi}{8} \left(\frac{\mathcal{L}(\mathcal{T})}{\mathcal{L}(\mathcal{N})} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} . \qquad (5.10)
\end{aligned}$$

[#11]A useful identity throughout here is

$$\sqrt{\mathcal{A}(\mathcal{T})}\sqrt{\mathcal{L}(\mathcal{T}/4)} = \mathcal{A}(\mathcal{T}/2)$$

We will see later how these cancel various one-loop contributions.

Extending this to Op^+ plane is immediate with

$$S_{Op^+} = -S_{Op^-} ,\qquad(5.11)$$

as motivated by the fact that the two planes differ by a sign of the charge. Again writing

$$\delta(S_{Op^+} + S_{Dp}) = (-1)^{(p+1)/2} \int (\Delta_{BB} + \Delta_{BO^++O^+B} + \Delta_{O^+O^+}) ,$$

the only change from O^- case is the sign flip of $\Delta_{BO^++O^+B} = -\Delta_{BO^-+O^-B}$. As such, we have

$$\begin{aligned}
&(-1)^{(p+1)/2} \left(\Delta_{BB} + \Delta_{BO^++O^+B}\right)\\
&= -(-1)^{(p+1)/2} \left(\frac{\pi}{2}\,[ch_{2k\otimes 2k}(\mathcal{F}) + ch_{2k}(2\mathcal{F})] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N})\right)^{(1)}_{p+1} ,
\end{aligned}\qquad(5.12)$$

where the trace in ch_ρ should be understood as taken in ρ representations of $Sp(k)$ gauge group. The defining representation $2k$ is pseudo-real, so the algebra goes the same as $SO(2k)$ cases. The Orientifold contribution

$$\Delta_{O^+O^+} = \Delta_{O^-O^-}\qquad(5.13)$$

remains the same, begins quadratic in the p-brane charge.

Inflow in the presence of $\widetilde{Op^-}$'s can be similarly obtained. Since the charge of $\widetilde{Op^-}$ equals to that of an Op^- plus an half D-brane, the obvious candidate for the CS coupling of $\widetilde{Op^-}$ is

$$S_{\widetilde{Op^-}} = \frac{1}{2} \times \left(-\frac{\mu_p}{2} \int \sum_r s^*(C_{r+1}) \wedge \left[\sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} - 2^{p-4}\sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}}\right]\right) .\qquad(5.14)$$

Δ_{BB} is unaffected as before, while $\Delta_{\widetilde{O^-}B+B\widetilde{O^-}}$ is modified as

$$\Delta_{B\widetilde{O^-}+\widetilde{O^-}B} = \frac{\pi}{2}\left([ch_{2k}(2\mathcal{F}) - 2ch_{2k}(\mathcal{F})] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N})\right)^{(1)}_{p+1} .\qquad(5.15)$$

Thus, the analog of (5.9) and (5.12) here is

$$-(-1)^{(p+1)/2}\left(\frac{\pi}{2}\,[ch_{2k\otimes 2k}(\mathcal{F}) - ch_{2k}(2\mathcal{F}) + 2ch_{2k}(\mathcal{F})] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N})\right)^{(1)}_{p+1} .\qquad(5.16)$$

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Finally, the purely Orientifold contribution may look more involved than before, but turns out to be the same:

$$
\begin{aligned}
\Delta_{\widetilde{O^-}\widetilde{O^-}} &= -\frac{\pi}{2}\left(\left(\sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} - 2^{p-4}\sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}}\right)^2\right)^{(1)}_{2p-6} \wedge \chi(\mathcal{N})_{9-p} \\
&= -\frac{\pi}{2}\left(\left(\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} - 2^{p-3}\frac{\mathcal{A}(\mathcal{T}/2)}{\mathcal{A}(\mathcal{N}/2)} + 2^{2(p-4)}\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}\right)\right)^{(1)}_{2p-6} \wedge \chi(\mathcal{N})_{9-p} \\
&\simeq -\frac{\pi}{2}\cdot 2^{2(p-4)}\left(\left(\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)} \wedge \chi(\mathcal{N})\right)\right)^{(1)}_{p+3} \\
&= -\frac{\pi}{8}\left(\frac{\mathcal{L}(\mathcal{T})}{\mathcal{L}(\mathcal{N})} \wedge \chi(\mathcal{N})\right)^{(1)}_{p+1} = \Delta_{O^-O^-} ,
\end{aligned}
\tag{5.17}
$$

where the equalities hold because we are supposed to extract $p+3$-form parts of the anomaly polynomial.

5.2 One-Loop from Open String Sector

Consider the situation where $2k$ coincident D-branes are on the top of one of an O^-, an O^+, or an $\widetilde{O^-}$ plane. There is one more type of Orientifold plane $\widetilde{O^+}$, but this leads to the same gauge group as the O^+ case and thus the same world-volume one-loop anomaly is induced.

First, in the presence of the O^- planes, the gauge group of the open strings ending on Dp-branes is enhanced from $U(k)$ to $SO(2k)$. Hence a $SO(2k)$ adjoint fermion contributes to the world-volume anomaly polynomial of amount

$$
2\pi \cdot ch_{\frac{1}{2}2k(2k-1)} \wedge \mathcal{A}(\mathcal{T}) \wedge ch_{S^+}(\mathcal{N})
\tag{5.18}
$$

for $4n$-dimensions, and

$$
\pi \cdot ch_{\frac{1}{2}2k(2k-1)} \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})]
\tag{5.19}
$$

for $4n+2$-dimensions. Thanks to the reality of $SO(2k)$, two of these can be written uniformly as

$$
I^{SO(2k)}_{1-loop} = \pi \cdot ch_{\frac{1}{2}2k(2k-1)} \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})] .
\tag{5.20}
$$

By the way, we have an identity

$$
ch_{\frac{1}{2}2k(2k\pm1)}(\mathcal{F}) = \frac{1}{2}ch_{2k\otimes2k}(\mathcal{F}) \pm \frac{1}{2}ch_{2k}(2\mathcal{F})
\tag{5.21}
$$

and it leads to

$$I_{1-loop}^{SO(2k)} = \frac{\pi}{2} \left[ch_{2k\otimes 2k} - ch_{2k}(2\mathcal{F}) \right] \wedge \mathcal{A}(\mathcal{T}) \wedge \left[ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N}) \right]. \quad (5.22)$$

Again, with the identity

$$\frac{\chi(\mathcal{N})}{\mathcal{A}(\mathcal{N})} = ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N}) , \quad (5.23)$$

we see that they have the precise form and the factor that can cancel inflows (5.9) from BB and $BO + OB$ intersection.

Similarly, the other cases follow. The symplectic case is

$$\begin{aligned} I_{1-loop}^{Sp(k)} &= \pi \cdot ch_{\frac{1}{2}2k(2k+1)} \wedge \mathcal{A}(\mathcal{T}) \wedge \left[ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N}) \right] \\ &= \frac{\pi}{2} \left[ch_{2k\otimes 2k}(\mathcal{F}) + ch_{2k}(2\mathcal{F}) \right] \wedge \mathcal{A}(\mathcal{T}) \wedge \left[ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N}) \right] , \end{aligned}$$

which are again canceled by the anomaly inflow $\Delta_{BB} + \Delta_{BO^+ + O^+B}$ (5.12) in the presence of an O^+ plane. $SO(2k + 1)$ type gauge theory can be also dealt with by expanding its adjoint representation in terms of the $SO(2k)$ representation as

$$ch_{adj.}^{SO(2k+1)} = ch_{\frac{1}{2}2k(2k-1)+2k} = \frac{1}{2} ch_{2k\otimes 2k}(\mathcal{F}) - \frac{1}{2} ch_{2k}(2\mathcal{F}) + ch_{2k}(\mathcal{F}) , \quad (5.24)$$

whereby the world-volume anomaly can be decomposed as

$$\begin{aligned} I_{1-loop}^{SO(2k+1)} &= \frac{\pi}{2} \left[ch_{2k\otimes 2k}(\mathcal{F}) - ch_{2k}(2\mathcal{F}) + 2ch_{2k}(\mathcal{F}) \right] \\ &\wedge \mathcal{A}(\mathcal{T}) \wedge \left[ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N}) \right] , \end{aligned} \quad (5.25)$$

which again is neatly canceled by $\Delta_{BB} + \Delta_{B\widetilde{O^-} + \widetilde{O^-}B}$ (5.16).

Hence, we conclude that the part of anomaly and inflow that depend on the gauge group exactly cancel regardless of the brane types, after the overall chirality (or the orientation issue) is properly taken into account.

5.3 On Universal Inflow Δ_{OO}

As $\Delta_{BB} + \Delta_{BO+OB}$ are canceled by the open string sector one-loop, Δ_{OO} is left uncanceled so far. Clearly this part of inflow has nothing to do with the open string degrees of freedom; it exists even in the absence of any D-branes. As such, Δ_{OO} should be canceled by one-loop anomaly from the closed string spectrum. We wish to emphasize here that, even before checking cancelation against closed string one-loop,

the proposed Chern-Simons couplings stand out because they lead to a universal inflow

$$\Delta_{O^-O^-} = \Delta_{O^+O^+} = \Delta_{\widetilde{O^-}\widetilde{O^-}} = -\frac{\pi}{8}\left(\frac{\mathcal{L}(\mathcal{T})}{\mathcal{L}(\mathcal{N})} \wedge \chi(\mathcal{N})\right)^{(1)}_{p+1}, \qquad (5.26)$$

from all types of Orientifold planes. This has to be the case, as the closed string part of the low energy spectrum does not care what kind of projections are taken on the Chan-Paton factors. This obvious and basic requirement is met by our Chern-Simons couplings, which may be compared to those in Refs. [13, 14, 15].

Checking the cancelation of Δ_{OO} by closed string one-loop for $p < 9$ is a bit nontrivial, however. The simplest thing to try would be the compact version of the same problem of T^{9-p}/Z_2 with 2^{9-p} Orientifold planes distributed, one at each fixed point. The low energy spectra here would be identical to type I theory compactified on T^{9-p}, producing one gravity multiplet and $(9-p)$ vector multiplets, transforming as vector representation under $SO(9-p)_R$. For $p = 5, 7$, in particular, one can see that the one-loop of this spectra does not completely cancel $2^{9-p}\Delta_{OO}$.[#12] That is, unless we set the normal bundle $\mathcal{N}$ to be trivial. In the latter case, both the inflow and the one-loop vanish individually.

In retrospect, this mismatch is to be expected since the one-loop computation based on the massless spectra in $p + 1$ dimensions only is really computing smeared version of the anomaly, over T^{9-p}, rather than the localized ones. As such, the normal bundle information, which measures nontrivial curvature effect along T^{9-p} direction to begin with, is inevitably lost along the way [29]. One must rely on more complete information, where higher modes such as Kaluza-Klein modes are taken into account, along the line of Ref. [30]. This is not an easy task, since one must also keep track of nontrivial internal curvatures. Instead we will consider $p = 9$ case that sidesteps

---

[#12]For $p = 3$, nevertheless, we do have a complete cancelation of Δ_{OO}

$$\int_{3+1} -2^6\frac{\pi}{8}\left[\frac{\mathcal{L}(\mathcal{T})}{\mathcal{L}(\mathcal{N})} \wedge \chi(\mathcal{N})\right]^{(1)} = -8\pi \int_{3+1} \chi(\mathcal{N})^{(1)}, \qquad (5.27)$$

by type I massless closed string spectra on T^6. The latter's one-loop gives

$$2\pi \cdot \mathcal{A}(\mathcal{T}) \wedge \left[ch_{S^-}(\mathcal{N}) + [ch_V(\mathcal{T}) - 1] \wedge ch_{S^+}(\mathcal{N}) + ch_{S^+}(\mathcal{N}) \wedge ch_V(\mathcal{N})\right]\Bigg|_{6-form}.$$

Since other factors involve only 4-forms or higher, we may replace $ch_{S^\pm}(\mathcal{N})$ by $\pm\chi(\mathcal{N})\mathcal{A}(\mathcal{N})^{-1}/2$. The Euler class $\chi(\mathcal{N})$ is 6-form, so the one-loop anomaly is

$$\int_{3+1}(-\pi + 3\pi + 6\pi) \cdot \chi(\mathcal{N})^{(1)} = 8\pi \int_{3+1} \chi(\mathcal{N})^{(1)}, \qquad (5.28)$$

canceling the inflow precisely.

this complication.

5.4 D9-O9$^-$: Green-Schwarz and Cancelation of Δ_{OO}

For O9$^-$ plane, this problem does not surface because a transverse direction does not exist. The cancelation between Δ_{OO} and closed string one-loop is really a well-known refinement of the type I theory Green-Schwarz mechanism, and is a standard material (e.g. see Ref. [31]). We record it here for the sake of completeness.

Consider the case of $2k$ D9 and a single O9$^-$ in the coverings space. Recall that the anomaly cancelation in type I theory involves two steps. The first is a tadpole condition $2k = 32$, leading to $SO(32)$, and the second is the Green-Schwarz mechanism generated by the coupling of type [32],

$$\sim \int C_2 \wedge X_8 \ , \tag{5.29}$$

which cancels via a modified Bianchi identity of $d(dC_2) = \cdots + X_4$ an anomaly of type

$$\sim X_4 \wedge X_8 \ . \tag{5.30}$$

As we have set up general anomaly inflow mechanism based on the Chern-Simons couplings to C's, we should be able to recast the Green-Schwarz mechanism in the current, more general framework [31].

First of all, recall that, among the RR tensor fields of type IIB theory, only C_2 and its dual C_6 survive the Orientifold projection to type I theory. The Chern-Simons coupling for $p = 9$ is

$$\begin{aligned} S_{CS} &= \frac{1}{2} \cdot \frac{\mu_9}{2} \int \left(\sum_r{}' C_{r+1} \wedge [Y(\mathcal{F}, \mathcal{R}) - 32Z(\mathcal{R})] \right) \tag{5.31} \\ &= \frac{1}{2} \cdot \frac{\mu_9}{2} \int C_2 \wedge [Y(\mathcal{F}, \mathcal{R}) - 32Z(\mathcal{R})]_8 + C_6 \wedge [Y(\mathcal{F}, \mathcal{R}) - 32Z(\mathcal{R})]_4 \ , \end{aligned}$$

where Y and Z are defined as (5.5). C_0, C_4, C_8 are projected out while C_{10} does not exist as a dynamical field, so that anomaly inflow has the polynomial,

$$-\frac{\pi}{2} \left([Y(\mathcal{F}, \mathcal{R}) - 32Z(\mathcal{R})]_4 + [Y(\mathcal{F}, \mathcal{R}) - 32Z(\mathcal{R})]_8 \right)^2 \ . \tag{5.32}$$

When expanded, the inflow can be also organized as

$$\Delta' = -\frac{\pi}{2} \left[ch_{2k \otimes 2k}(\mathcal{F}) \wedge \mathcal{A}(\mathcal{R}) \right]_{10}^{'(1)} + \frac{\pi}{2} \left[ch_{2k}(2\mathcal{F}) \wedge \mathcal{A}(\mathcal{R}) \right]_{10}^{'(1)} - \frac{\pi}{8} [\mathcal{L}(\mathcal{R})]_{10}^{'(1)} \ , \tag{5.33}$$

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where each terms are from BB, $BO+OB$ and OO intersections respectively, and the prime $'$ signifies that we dropped terms proportional to Y_0 and Z_0 when expanding $(Y-32Z)^2$ to compute Δ.

On the other hand, from the supergravity multiplet, we have a left-handed gravitino and a right-handed dilatino which are both Majorana-Weyl. They carry 12-form anomaly polyniomial of amount

$$I_{closed} = \frac{2\pi}{2} \left[I_{3/2}(R) - I_{1/2}(R) \right]'_{12} = \frac{\pi}{8} \cdot \mathcal{L}(\mathcal{R})_{12} . \tag{5.34}$$

In the open string sector, there are Majorana-Weyl gauginos in the adjoint representation of $SO(2k)$. They contribute 12-form of

$$
\begin{aligned}
I_{open} &= \frac{2\pi}{2} \cdot \left[\mathcal{A}(\mathcal{R}) \wedge ch_{\frac{2k(2k-1)}{2}}(\mathcal{F}) \right]_{12} \\
&= \frac{\pi}{2} \cdot \left[\mathcal{A}(\mathcal{R}) \wedge ch_{2k \otimes 2k}(\mathcal{F}) - \mathcal{A}(\mathcal{R}) \wedge ch_{(2k)}(2\mathcal{F}) \right]_{12} . \tag{5.35}
\end{aligned}
$$

These two can be combined,

$$I_{closed} + I_{open} = \frac{\pi}{2} \left[\left(Y(\mathcal{F},\mathcal{R}) - 32Z(\mathcal{R}) \right)^2 \right]_{12} , \tag{5.36}$$

which superficially looks similar to the inflow up to sign.

Note that this one-loop anomaly does not match the inflow above. The inflow cancels

$$\pi \left[Y(\mathcal{F},\mathcal{R}) - 32Z(\mathcal{R}) \right]_4 \wedge \left[Y(\mathcal{F},\mathcal{R}) - 32Z(\mathcal{R}) \right]_8 , \tag{5.37}$$

but the other terms in (5.36) remain. Of these, $([Y(\mathcal{F},\mathcal{R})-32Z(\mathcal{R})]_6)^2$ piece vanishes identically on its own, so the discrepancy is

$$\pi \left[Y(\mathcal{F},\mathcal{R}) - 32Z(\mathcal{R}) \right]_0 \wedge \left[Y(\mathcal{F},\mathcal{R}) - 32Z(\mathcal{R}) \right]_{12} , \tag{5.38}$$

bringing us to the usual tadpole condition of type I theory

$$Y_0(\mathcal{F},\mathcal{R}) - 32Z_0(\mathcal{R}) = 2k - 32 = 0 , \tag{5.39}$$

for a complete anomaly cancelation.

With this tadpole condition obeyed, the closed string sector one-loop I_{closed} cancels on its own against the purely Orientifold contribution,

$$\Delta'_{OO} = -(\pi/8)[\mathcal{L}(\mathcal{R})]'^{(1)}_{10} = -2^9 \pi [\mathcal{L}(\mathcal{R})/4]'^{(1)}_{10} = -2^9 \pi [Z(\mathcal{R})^2]'^{(1)}_{10} , \tag{5.40}$$

and the open string one-loop I_{open} cancels the rest of the inflow, $\Delta' - \Delta'_{OO}$, coming from BB and $BO + OB$ intersections.

6   Summary

In this note, we re-examined in detail the inflow mechanism onto the various world-volume theories, with the aim to clear up loose ends on I-brane/D-brane inflow.

We started with a review of the M5-brane inflow and the I-brane/D-brane inflow. Historically, the M5-brane anomaly inflow was studied in two steps. Some anomaly inflow onto the M5 arises rather straightforwardly out of a Chern-Simons coupling between antisymmetric tensor field C_3 and spacetime curvature, but this turned out inadequate for the $SO(5)_R$ axial anomaly, as was first noted by Witten (2.12). For the I-brane/D-brane anomaly, the necessary topological coupling lives on the world-volume, rather than on spacetime, and is linear in the RR antisymmetric tensor fields. The difference from the M5 example is that many RR fields enter the inflow mechanism simultaneously. For the latter also, the cancelation against one-loop was partial in that there were, so-called self-dual cases, such as D3-branes, where the necessary inflow were apparently absent.

The problem of the axial anomaly deficit for the M5-brane was eventually solved by FHMM, who noted that one should carefully treat the singularity at the position of the M5-brane. They replaced the usual naive delta-function by a covariantized and smeared version, and at the same time demanded a regularity of the resulting field strengths. It requires a particular transformation rule for the three-form gauge field, C_3, and as a consequence, they yield a right inflow from Chern-Simon terms of type, $C_3 \wedge dC_3 \wedge dC_3$.

It is then almost immediate that the regularity requirement on the field strengths should be also obeyed in the I-brane/D-brane inflow mechanism. With the Thom class for IIB D-branes, $\tau_n = d[\rho \cdot \hat{e}_{n-1}] = d[\rho \cdot (d\psi_{n-2} + \Omega_{n-1})]$, we should choose its descent as

$$\tau_{n-1}^{(0)} = -d\rho \wedge \psi_{n-2} + \rho \cdot \Omega_{n-1} \ , \tag{6.1}$$

when solving the Bianchi identity. Note that this choice of $\tau_{n-1}^{(0)}$ has a nontrivial transformation under the normal bundle gauge transformation, unlike the naive, invariant, but singular choice $\rho \cdot \hat{e}_{n-1}$. This modifies the variation of the RR tensor fields (4.13), and results in a new form of inflow (4.16). For $p \geq 5$, the new inflow agrees with the old one, up to local counter-terms on world-volume, while for $p = 3$ the inflow no longer vanishes and neatly cancels the one-loop anomaly of the D3-brane open string sector.

Furthermore, the modified inflow mechanism is such that the correct answer emerges regardless of specific form of the Chern-Simons couplings used, be it S_{CS} (1.3) or S'_{CS} (2.15),

$$\delta S_{CS} \simeq \delta S'_{CS} \ , \tag{6.2}$$

where the difference between the two amounts to a local counter-term on the world-volume. This should be contrasted to the previous inflow mechanism that resulted in correct answer only from S'_{CS}. With old $\tilde{\delta}$ of (2.23), $\tilde{\delta}S'_{CS}$ is NOT equivalent to $\tilde{\delta}S_{CS}$ up to a local counter-term, even when it produces correct inflow. This was a little curious and potentially confusing, all the more, as S_{CS} looks by far more natural than S'_{CS} yet failed to generate the anticipated inflow via $\tilde{\delta}$. By the physically motivating revised transformation rule δ (4.13), instead, we also cured this phenomenon.

Finally, we extended this to the theories including the Orientifold planes. Curiously, there appears to be no complete consensus on the gravitational Chern-Simons couplings on some of Orientifold planes. We settled this by requiring cancelation between anomaly inflow and one-loop contributions from the open string sector and the closed string sector. We computed the most general one-loop anomaly of the maximally supersymmetric Yang-Mills theories in all even dimensions, and determined the necessary Chern-Simons couplings on the four types of Orentifold planes as in (5.1), (5.2) and (5.3). Gauge theory one-loop anomaly completely cancels out part of the inflow that involves D-branes. The pure Orientifold part of inflow, Δ_{OO}, to be canceled by the closed string sector one-loop, is also shown to be universal, i.e., independent of types of the Orientifold, which, we argued, is by itself a nontrivial consistency check. Our result for O^+ planes, in particular, agrees with Refs. [11][12].

Acknowledgments

A $1/2$ in the Minimal Couplings

In this appendix, we show the simplest example of duality symmetric formulation of p-form theory and illustrate how the additional factor $1/2$ in the minimal coupling is necessary. For more comprehensive studies, see Refs. [33][19]. The simplest example

is a Maxwell theory in $d = 4$, whose electric form is

$$\frac{1}{2} \int dx^4 \, (\mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B}) - e \int dt \, (\mathbf{A} \cdot \dot{\mathbf{r}} + \phi) \ , \tag{A.1}$$

with $A_0 = \phi$ and electric charge e. Coupling this to both electric and magnetic charges simultaneously is more involved but can be done relatively easily at the expense of explicit Lorentz invariance. For this, one introduces two sets of Maxwell fields,

$$\mathbf{B}^{(a)} = \nabla \times \mathbf{A}^{(a)} + \cdots \ , \qquad \mathbf{E}^{(a)} = \partial_t \mathbf{A}^{(a)} - \nabla \phi^{(a)} + \cdots \ , \tag{A.2}$$

for $a = 1, 2$, where the ellipses encode possible violations of the Bianchi identities, and invents a duality symmetric action,

$$\frac{1}{2} \int dx^4 \left(\mathbf{B}^{(1)} \cdot \mathbf{E}^{(2)} - \mathbf{B}^{(2)} \cdot \mathbf{E}^{(1)} - \mathbf{B}^{(1)} \cdot \mathbf{B}^{(1)} - \mathbf{B}^{(2)} \cdot \mathbf{B}^{(2)} \right)$$
$$- \frac{1}{2} \, q^{(2)} \int dt \left(\mathbf{A}^{(1)} \cdot \dot{\mathbf{r}} + \phi^{(1)} \right) + \frac{1}{2} \, q^{(1)} \int dt \left(\mathbf{A}^{(2)} \cdot \dot{\mathbf{r}} + \phi^{(2)} \right) \ . \tag{A.3}$$

The Gauss constraints are

$$\nabla \cdot \mathbf{B}^{(a)} + q^{(a)} \delta(\mathbf{x} - \mathbf{r}(t)) = 0 \ , \tag{A.4}$$

which also imply

$$\partial_t \mathbf{B}^{(a)} - \nabla \times \mathbf{E}^{(a)} - q^{(a)} \dot{\mathbf{r}} \, \delta(\mathbf{x} - \mathbf{r}(t)) = 0 \ . \tag{A.5}$$

Note that $q^{(1)}$ and $q^{(2)}$ act like magnetic charges of $-\mathbf{B}^{(1)}$ and $-\mathbf{B}^{(2)}$, although they will be eventually identified as electric charges of $-\mathbf{E}^{(2)}$ and $\mathbf{E}^{(1)}$ later.

Consider the simple case of $(q^{(1)}, q^{(2)}) = (0, e)$. With $q^{(1)} = 0$, equation of motion for $\mathbf{A}^{(2)}$ is solved as

$$\mathbf{E}^{(1)} + \mathbf{B}^{(2)} = 0 \ , \tag{A.6}$$

killing off $-\mathbf{B}^{(2)} \cdot \mathbf{E}^{(1)} - \mathbf{B}^{(2)} \cdot \mathbf{B}^{(2)}$ in the action, and one of the Gauss constraints (A.4) becomes more conventional,

$$\nabla \cdot \mathbf{E}^{(1)} = e \, \delta(\mathbf{x} - \mathbf{r}(t)) \ . \tag{A.7}$$

The remaining terms in the action are

$$\frac{1}{2} \int dx^4 \left(\mathbf{B}^{(1)} \cdot \mathbf{E}^{(2)} - \mathbf{B}^{(1)} \cdot \mathbf{B}^{(1)} \right) - \frac{1}{2} e \int dt \left(\mathbf{A}^{(1)} \cdot \dot{\mathbf{r}} + \phi^{(1)} \right)$$
$$= \frac{1}{2} \int dx^4 \left(\mathbf{A}^{(1)} \cdot \nabla \times \mathbf{E}^{(2)} - \mathbf{B}^{(1)} \cdot \mathbf{B}^{(1)} \right) - \frac{1}{2} e \int dt \left(\mathbf{A}^{(1)} \cdot \dot{\mathbf{r}} + \phi^{(1)} \right)$$
$$= \frac{1}{2} \int dx^4 \left(-\mathbf{A}^{(1)} \cdot \partial_t \mathbf{E}^{(1)} - \mathbf{B}^{(1)} \cdot \mathbf{B}^{(1)} \right) - \frac{1}{2} e \int dt \left(2\mathbf{A}^{(1)} \cdot \dot{\mathbf{r}} + \phi^{(1)} \right) \ , \tag{A.8}$$

where we used (A.5) and (A.6). With $q^{(1)} = 0$, the Bianchi identity of the first gauge field holds, so we may use $\partial_t \mathbf{A}^{(1)} = \mathbf{E}^{(1)} + \nabla \phi^{(1)}$. Integrating by parts and using the Gauss constraint (A.4) again,

$$\frac{1}{2} \int dx^4 \left(\mathbf{E}^{(1)} \cdot \mathbf{E}^{(1)} - \mathbf{B}^{(1)} \cdot \mathbf{B}^{(1)} \right) - e \int dt \left(\mathbf{A}^{(1)} \cdot \dot{\mathbf{r}} + \phi^{(1)} \right) , \qquad \text{(A.9)}$$

we find the usual Maxwell action with electric charge e, without the factor $1/2$. This shows that the correct equation of motion emerges, even though the minimal coupling has an unfamiliar factor $1/2$.

One may repeat the exercise, with a static charge $(q^{(1)}, q^{(2)}) = (-g, 0)$ instead, by integrating out the other Maxwell fields. The equation of motion from $\mathbf{A}^{(1)}$ is solved as

$$\mathbf{E}^{(2)} - \mathbf{B}^{(1)} = 0 , \qquad \text{(A.10)}$$

and, a similar procedure produces

$$\frac{1}{2} \int dx^4 \left(\mathbf{E}^{(2)} \cdot \mathbf{E}^{(2)} - \mathbf{B}^{(2)} \cdot \mathbf{B}^{(2)} \right) - g \int dt \left(\mathbf{A}^{(2)} \cdot \dot{\mathbf{r}} + \phi^{(2)} \right) , \qquad \text{(A.11)}$$

again without the factor $1/2$.

The factor $1/2$ in the symmetric formulation is also consistent with the Dirac quantization. Suppose that a particle with $(q^{(1)}, q^{(2)}) = (0, e)$ is present in the vicinity of another with $(q^{(1)}, q^{(2)}) = (-g, 0)$. In usual electric formulation, where e and g are electric and magnetic charges of $\mathbf{A}$, the quantization comes from the invisibility of the Dirac string of the latter as the former circles it along a small loop γ. The phase shift on the former wavefunction would be

$$e \int_\gamma \mathbf{A}_g \cdot d\mathbf{x} = 4\pi \, e \cdot g . \qquad \text{(A.12)}$$

In the duality symmetric formulation, however, both particles generate Dirac strings, due to (A.4), and we have two such contributions to the phase shift. For simplicity, we may imagine the two Dirac strings stretched along positive and negative z-axis, respectively. As the first particle moves along γ, encircling a Dirac string, the second particle also circles around a Dirac string of the other, along $-\gamma$, once. The combined phase shift is

$$\frac{1}{2} e \int_\gamma \mathbf{A}_g^{(1)} \cdot d\mathbf{x} + \frac{1}{2} g \int_{-\gamma} \mathbf{A}_{-e}^{(2)} \cdot d\mathbf{x} = 2\pi \, e \cdot g + 2\pi g \cdot e = 4\pi \, e \cdot g . \qquad \text{(A.13)}$$

Thus, the factor $1/2$ is not only consistent with but necessary for preserving the usual Dirac quantization condition.

B Duality Symmetric Action for RR Fields

What we saw in the previous section extends to the collection of RR tensors in type II theories as

$$\frac{1}{4\kappa_{10}^2} \int d^{10}x \sum_n \Big[B_n \wedge *(B_n) - (-1)^{(n+\epsilon-2)/2} B_n \wedge E_{10-n} \Big]$$
$$+\frac{1}{2} \sum_p \mu_p \int_{Dp} \sum_q s^*(C_{q+1}) \wedge Y_{p-q} \ , \tag{B.1}$$

with the universal $1/2$ factor in the minimal coupling. The Hodge star operation is taken to act on the right,

$$* B_{i_{n+1}\cdots i_d} = \frac{1}{n!} B_{i_1\cdots i_n} \epsilon^{i_1\cdots i_n}{}_{i_{n+1}\cdots i_d} \ , \tag{B.2}$$

so that the first term is negative definite with $(-++\cdots+)$ signature.[13] As in Appendix A, $H_n = dC_{n-1}$ is split into space-like B and time-like E. More precisely, if we split $C = \mathbf{C} + \Phi$ with the time-like part Φ and similarly $d = \mathbf{d} + d_t$, we have

$$B = \mathbf{d}\mathbf{C} + \cdots \ , \qquad E = d_t\mathbf{C} + \mathbf{d}\Phi + \cdots \ , \tag{B.3}$$

again up to terms in the ellipses that violate the naive Bianchi identity.

Note that, using the same line of argument as in Appendix A, we obtain $*H_n = (-1)^{(n-2+\epsilon)/2}H_{10-n}$, and thus the first line is the duality symmetric kinetic term that we implicitly used in this note. The field equation and the Bianchi identity are

$$d(*(H_{r+2})) = -(-1)^r 2\kappa_{10}^2 \mu_r \tau_{9-r} \ , \qquad dH_{r+2} = (-1)^{(r+\epsilon)/2} 2\kappa_{10}^2 \mu_{6-r} \tau_{r+3} \ , \tag{B.4}$$

with a single Dr-brane and with a single D$(6-r)$-brane, respectively, and all curvatures turned off. More generally, the Bianchi identity with curvatures turned on and other D-branes present is

$$dH_{r+2} = \sum_p (-1)^{(r+\epsilon)/2} 2\kappa_{10}^2 \mu_p \, Y_{p+r-6} \wedge \tau_{9-p} = -\sum_p (-1)^{(-p+\epsilon)} 2\kappa_{10}^2 \mu_p \, \bar{Y}_{p+r-6} \wedge \tau_{9-p} \ , \tag{B.5}$$

where the right hand side represents induced D$(6-r)$-brane charges on Dp-branes.

For 4-form field C_4, with its dual also a 4-form, we need to be more careful. H_5 is constrained to be self-dual, so degrees of freedom counting suggests only one

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[13]This type of duality symmetric action for tensor fields works in $d = 4k+2$ with (anti-)self-dual middle form, while, for $d = 4k$, the middle form must be doubled as in Appendix A and the sign for half of kinetic terms must be flipped.

C_4 enter the action. Indeed, it is known [19] that such a kinetic term for a single C_4 generates self-duality constraint from equation of motion. Thus, the pertinent question is whether the single minimal coupling of D3 to C_4 is also consistent with self-duality of H_5 and whether the same factor of $1/2$ in the minimal coupling leads to correctly Dirac-quantized sources. Here, we will show that the homogeneous and the inhomogeneous part of H_5 are respectively self-dual, as a consequence of the above kinetic term.

First, we review the source-free case for $n = 5$ and $\epsilon = 1$. With the spatial indices denoted by capital roman characters, A, B, etc, note that

$$H_{ABCDE} = \partial_A C_{BCDE} + \cdots + \partial_E C_{ABCD} \ , \tag{B.6}$$

where the sum is over the cyclic permutations. In the absence of source, the action (B.1) reduces

$$I = \frac{1}{2} \int d^{10}x \left[E^{ABCD} B_{ABCD} - B^{ABCD} B_{ABCD} \right] \ , \tag{B.7}$$

where we define B and E by the magnetic and electric components of H,

$$B_{ABCD} = \frac{1}{120} \epsilon^{ABCDEFGHI} H_{EFGHI} \ , \tag{B.8}$$

$$E_{ABCD} = -H^{0ABCD} \ , \tag{B.9}$$

with the indices $A \sim I = 1 \cdots 9$. Then variation of (B.7) with C_{ABCD} gives

$$\frac{1}{2} \left[\frac{1}{12} \epsilon^{FGHIEABCD} \partial_E \left(E^{FGHI} - B^{FGHI} \right) \right] = 0 \ , \tag{B.10}$$

and we can choose C_{0ABC} so that the solution can be written as

$$E^{FGHI} = B^{FGHI} \ , \tag{B.11}$$

which is equivalent to $H_5 = *H_5$, the self-duality equation.

The relation (B.11) also holds in the presence of a source term. Now the action is written in a form

$$
\begin{aligned}
I = \ & \frac{1}{2} \int d^{10}x \left[E^{ABCD} B_{ABCD} - B^{ABCD} B_{ABCD} \right] \\
& + \frac{1}{2} \int d^{10}x \left[C_{ABCD} J^{ABCD} + C_{0ABC} J^{0ABC} \right] \ ,
\end{aligned} \tag{B.12}
$$

where J^{ABCD} is a current source. We also add possible contributions of source to the field strengths by

$$B_{ABCD} = \frac{1}{120} \epsilon^{ABCDEFGHI} H_{EFGHI} - G^{0ABCD} \ , \tag{B.13}$$

$$E_{ABCD} = -H^{0ABCD} + F^{ABCD} \ . \tag{B.14}$$

Here, the additional term G can be thought of as the inhomogeneous solution,

$$\partial_E G^{ABCDE} - \partial_0 G^{0ABCD} = -J^{ABCD} \ , \tag{B.15}$$

which is consistent with the Bianchi identity

$$\partial_D B^{ABCD} = -\partial_D G^{0ABCD} = -J^{0ABC} \ . \tag{B.16}$$

Finally, variation of the action with C_{ABCD} gives an equation of motion,

$$\frac{1}{12}\epsilon^{FGHIEABCD}\partial_E \left(B_{FGHI} - E_{FGHI}\right)$$
$$-\partial_0 G^{0ABCD} + \frac{1}{24}\epsilon^{ABCDEFGHI}\partial_E F_{FGHI} + J^{ABCD} = 0 \ . \tag{B.17}$$

Thanks to the self-duality relation in the absence of source and (B.15), we can conclude from the equation of motion that F^{ABCD} should satisfy

$$F^{ABCD} = \frac{1}{120}\epsilon^{ABCDEFGHI}G_{EFGHI} \ . \tag{B.18}$$

Note that this equation implies the source term contributions, G and F of (B.13) and (B.14), also should be self-dual, which requires the dyonic source of the equal magnetic and electric charge. Then again by a suitable choice of C_{0ABC}, we can see that the self-duality equation,

$$B^{ABCD} = E^{ABCD} \ , \tag{B.19}$$

holds in general, even in the presence of the self-dual dyonic sources. Finally, combining (B.16) and (B.19), the correct field equations for C_4 with a dyonic source are induced, justifying the minimal coupling to C_4 only with the by-now-familiar factor $1/2$.

One consistency check is again the Dirac quantization condition. When one D3 revolves around another's Dirac-string-like singularity, we find the phase picked up in the process is

$$2 \times \frac{\mu_3}{2} \times 2\kappa_{10}^2\mu_3 = 2\kappa_{10}^2\mu_3^2 = 2\pi \ , \tag{B.20}$$

where the overall factor 2 occurs because each D3 acts as a magnetic source for the other's electric charge. This again makes the Dirac-string-like singularities invisible. This is not much different from other dual pairs, where 2π is achieved as

$$\frac{\mu_p}{2} \times 2\kappa_{10}^2\mu_{6-p} + \frac{\mu_{6-p}}{2} \times 2\kappa_{10}^2\mu_p = 2\kappa_{10}^2\mu_p\mu_{6-p} = 2\pi \ , \tag{B.21}$$

instead, for $p \neq 3$, again in the duality symmetric formulation.

C Characteristic Classes: Brief Summary

We list characteristic classes that appear in the anomaly inflow consideration. The Chern class is

$$ch_{\mathbf{R}}(\mathcal{F}) \equiv \mathrm{tr}_{\mathbf{R}} e^{\mathcal{F}/2\pi} = \sum_i e^{x_i} \ , \tag{C.1}$$

where $\mathbf{R}$ denotes the relevant representation, and x_i are the two-form-valued eigenvalues of $\mathcal{F}/2\pi$ in the representation $\mathbf{R}$. The A-roof genus and the Hirzbruch class for SO bundle are, in terms of skew-eigenvalue 2-forms y_i of $R/2\pi$,

$$\mathcal{A}(R) \equiv \prod_i \frac{y_i/2}{\sinh(y_i/2)} \ , \qquad \mathcal{L}(R) \equiv \prod_i \frac{y_i}{\tanh(y_i)} \ . \tag{C.2}$$

These can also be expanded in term of Pontryagin classes,

$$p_1(R) = \sum_i y_i^2 \ , \quad p_2(R) = \sum_{i<k} y_i^2 y_k^2 \ , \quad p_3(R) = \sum_{i<k<l} y_i^2 y_k^2 y_l^2 \ , \tag{C.3}$$

and so on. Finally, the Euler class is

$$\chi(R) = \prod_i y_i \ . \tag{C.4}$$

With these, we see

$$\frac{\chi(R)}{\mathcal{A}(R)} = \prod_i \frac{\sinh(y_i/2)}{y_i/2} \prod_j y_j = \prod_i \left(e^{y_i/2} - e^{-y_i/2} \right) = ch_{S^+}(R) - ch_{S^-}(R) \ , \tag{C.5}$$

for example, giving us the central identities in relating the inflow to the one-loop contribution, and also

$$\mathcal{A}(R)\mathcal{L}(R/4) = \prod_i \frac{2(y_i/4)^2}{\sinh(y_i/2)\tanh(y_i/4)} = \prod_i \frac{(y_i/4)^2}{\sinh(y_i/4)^2} = \mathcal{A}(R/2)^2 \ , \tag{C.6}$$

which was useful in section 5.

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